Low-Order Perturbation Analysis of a Multi-Country Complete Markets Model

Robert Kollmann [(*)]
ECARES, Université Libre de Bruxelles; Université Paris XII; and CEPR
50 Av. F.D. Roosevelt; B1050 Brussels; Belgium; robert_kollmann@yahoo.com

Sunghyun Henry Kim
Department of Economics, Suffolk University
8 Ashburton Place, Boston, MA 02108, USA; henry.kim@suffolk.edu

Jinill Kim
Federal Reserve Board, Mail Stop 70; 20th and Constitution Avenue, NW
Washington, DC 20551, USA; jinill.kim@frb.gov

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This paper solves the multi-country RBC model with complete markets defined in "Problem A" of den Haan, Judd and Juillard (2007), using the Sims (2000) algorithm that is based on second-order Taylor expansions of the equilibrium conditions. The algorithm is markedly more accurate than linear approximations, especially when the model exhibits strong curvature.

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[(*)] Corresponding author.

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1. Introduction

This paper solves the multi-country RBC model described in "Problem A" of the JEDC Numerical Methods Comparison Project (den Haan, Judd and Juillard (2007) [DJJ]) using Chris Sims' MATLAB program gensys2 which implements the Sims (2000) algorithm that is based on second-order Taylor expansions of the equilibrium conditions. Other solutions techniques for dynamic models based on expansions of second (or higher) order have been developed by Guu and Judd (1993), Gaspar and Judd (1996), Judd (1998), Jin and Judd (2004), Collard and Juillard (2001), Juillard (2004), Kim and Kim (2003), Schmitt-Grohé and Uribe (2004), Schaumburg (2002), Sutherland (2002), Kollmann (2003b) and Lombardo and Sutherland (2004).

Compared to other non-linear techniques (see Judd (1998)), second-order approximations have two key advantages: the ease with which they can be applied to models with a large number of state variables, and their high computational speed. This explains why a rapidly growing number of studies apply second-order accurate solution algorithms. Sims' gensys2 program is often used in these studies; see, e.g., Bergin and Tchakarov (2003), Kim and Kim (2001), Kim (2004), Kollmann (2002, 2003a, 2004, 2007), Marzo (2004), Shin (2004), Straub and Tchakarov (2005) and Teo (2003).

So far, there has been little systematic evaluation of the accuracy of solution methods based on second-order approximations. Together with other studies in the JEDC Numerical Methods Comparison Project, this paper fills that gap. Its main finding is that (for DJJ Problem A), the second-order solution based on the Sims algorithm can be markedly more accurate than linear approximations, especially when the model exhibits strong curvature.

2. Equilibrium

In normalized form, the model is defined by the following equations, for countries $n=1,\ldots,N$:

$$(\tau^n u^n_{c,t} - \lambda_t)/(\tau^n u^n_{c,t}) = 0\,, \tag{1}$$

$$(\lambda_t a^n_t f^n_{l,t} + \tau^n u^n_{l,t})/(\tau^n u^n_{l,t}) = 0\,, \tag{2}$$

$$E_t\{\beta\,(\lambda^n_{t+1}/\lambda^n_t)[a^n_{t+1} f^n_{k,t+1} + 1 + \varphi[1+\tfrac{1}{2}(i^n_{t+1}/k^n_{t+1}-\delta)](i^n_{t+1}/k^n_{t+1}-\delta)]/[1+\varphi(i^n_t/k^n_t-\delta)]\} - 1 = 0\,, \tag{3}$$

$$\left(\sum\nolimits_{n=1}^{N} c^n_t + i^n_t - \delta k^n_t - a^n_t f^n(k^n_t, l^n_t) + (\varphi/2)k^n_t(i^n_t/k^n_t-\delta)^2\right)\Big/\left(\sum\nolimits_{n=1}^{N} c^n_t + i^n_t - \delta k^n_t\right) = 0\,. \tag{4}$$

$$(k^n_{t+1} - i^n_t - (1-\delta)k^n_t)/k^n_t = 0 \tag{5}$$

$$\{a^n_t - \exp[\rho\ln(a^n_{t-1}) + \sigma(e_t + e^n_t)]\}/a^n_t = 0\,, \tag{6}$$

with $e_{t+1}, e^n_{t+1} \sim N(0,1)$, $u^n_{c,t} \equiv \partial u^n(c^n_t, l^n_t)/\partial c^n_t$, $u^n_{l,t} \equiv \partial u^n(c^n_t, l^n_t)/\partial l^n_t$, $f^n_{l,t} \equiv \partial f^n(k^n_t, l^n_t)/\partial l^n_t$, $f^n_{k,t} \equiv \partial f^n(k^n_t, l^n_t)/\partial k^n_t$.

3. Solution method [1]

The Sims (2000) algorithm can be applied to models of the following form:

$$E_t G(\underset{q\times1}{\omega_{t+1}}, \underset{q\times1}{\omega_t}, \underset{m\times1}{\varepsilon_{t+1}}) = 0, \tag{7}$$

where ω_t is a vector of variables known at date t, ε_{t+1} is a vector of exogenous disturbances with $E_t\varepsilon_{t+1} = 0$ and $E_t\varepsilon_{t+1}\varepsilon'_{t+1} = \Omega$. Sims (2000) assumes that the solution of (7) is unique and of the form

$$y_{t+1} = F(y_t, \varepsilon_{t+1})\,,\ x_{t+1} = M(y_{t+1})\,, \tag{8}$$

where y_t and x_t are linear functions of ω_t: $(y'_t\ x'_t)' = Z\omega_t$, for some square, non-singular matrix Z. Note that the solution can be expressed as

[1] For a more detailed presentation of the algorithm, see Kim, Kim, Schaumburg and Sims (2003).

$$\omega_{t+1} = \Psi(\omega_t, \varepsilon_{t+1}) \equiv Z^{-1}\begin{bmatrix} F(Z_1\omega_t, \varepsilon_{t+1}) \\ M(F(Z_1\omega_t, \varepsilon_{t+1})) \end{bmatrix}, \tag{9}$$

where Z_1 is the matrix (consisting of the first rows of *Z*) such that $y_t = Z_1\omega_t$.

Sims (2000) presents an algorithm (and a MATLAB program, gensys2) that constructs 2nd degree polynomials which approximate (8), in the neighborhood of a deterministic steady state given by $G(\omega,\omega,0)=0$. The coefficients of those polynomials are functions of Ω and of the first and second derivatives of $G(\omega_{t+1}, \omega_t, \varepsilon_{t+1})$ (evaluated at the steady state). Let $y_{t+1}=\widehat{F}(y_t,\varepsilon_{t+1})$, $x_{t+1}=\widehat{M}(y_{t+1})$ denote the polynomials that approximate (8), and

$$\omega_{t+1} = \widehat{\Psi}(\omega_t, \varepsilon_{t+1}) \equiv Z^{-1}\begin{bmatrix} \widehat{F}(Z_1\omega_t, \varepsilon_{t+1}) \\ \widehat{M}(\widehat{F}(Z_1\omega_t, \varepsilon_{t+1})) \end{bmatrix}. \tag{10}$$

Application to Problem A

(1)-(6) can be written like (7), using $\omega_t = (\ln(\lambda_t), \ln(c_t^1), .., \ln(c_t^N); \ln(l_t^1), .., \ln(l_t^N); \ln(i_t^1), .., \ln(i_t^n); \ln(k_{t+1}^1), .., \ln(k_{t+1}^N);$ $\ln(a_t^1), .., \ln(a_t^N))$; $\varepsilon_{t+1} \equiv (e_{t+1}; e_{t+1}^1, .., e_{t+1}^N)$; $\Omega \equiv \sigma^2 I_{1+N}$ (I_{1+N}: identity matrix with $1+N$ elements).

We use a two-point finite difference procedure (Fackler and Miranda (2002); pp.98, 102) to compute the first and second derivatives of $G(\omega_{t+1}, \omega_t, \varepsilon_{t+1})$ (at the steady state).

The accuracy checks discussed below require to formulate the solution as a "policy function" that expresses the date $t+1$ decision variables as a function of the capital stocks at the beginning of $t+1$, and of productivity at $t+1$ (in the N countries). Let $K_t \equiv (\ln(k_t^1), .., \ln(k_t^N))$, $A_t \equiv (\ln(a_t^1), .., \ln(a_t^N))$, $S_t \equiv (\ln(\lambda_t), \ln(c_t^1), .., \ln(c_t^N); \ln(l_t^1), .., \ln(l_t^N); \ln(i_t^1), .., \ln(i_t^n))$. As $\omega_t = (S_t, K_{t+1}, \mathrm{A}_t)$, the solution (10) can be written as: $\omega_{t+1} = \widehat{\Psi}((S_t, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1})$. For the model here, we verified that (10) has these properties: (i) the lagged "jump variables" S_t have zero influence on ω_{t+1}; and (ii) the influence of A_t and ε_{t+1} on ω_{t+1} can be subsumed by A_{t+1}. Thus, the approximate solution can be written as the following policy function: $\omega_{t+1} = \widehat{\Xi}(K_{t+1}, \mathrm{A}_{t+1})$. [2]

4. Accuracy

Let Ξ be the exact policy function that is approximated by $\widehat{\Xi}$. Ψ and Ξ satisfy the condition $E_t G(\Psi(\Xi(K_t, A_t), \varepsilon_{t+1}), \Xi(K_t, A_t), \varepsilon_{t+1}) = 0$. The accuracy tests evaluate how closely the approximate solution $\widehat{\Psi}, \widehat{\Xi}$ meets this condition. Let

$$\widehat{R}_t(K_t, A_t) \equiv E_t G(\widehat{\Psi}(\widehat{\Xi}(K_t, A_t), \varepsilon_{t+1}), \widehat{\Xi}(K_t, A_t), \varepsilon_{t+1}) \tag{11}$$

be the "conditional error function" of $\widehat{\Psi}, \widehat{\Xi}$. $\widehat{R}_t(K_t, A_t)$ is a vector with $5N+1$ elements (the model has $5N+1$ equations); let $\widehat{R}_{i,t}(K_t, A_t)$ be the i-th element of $\widehat{R}_t(K_t, A_t)$. We compute the expectation in (11) using the monomial integration formula of degree 3 in Judd (1998, p.275).

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[2] From (i): $\widehat{\Psi}((0, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1}) = \widehat{\Psi}((S_t, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1})$. From (6): $A_{t+1} = \rho A_t + \Lambda\varepsilon_{t+1}$ (Λ: a matrix). It appears that $\widehat{\Psi}((0, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1}) = \widehat{\Psi}((0, K_{t+1}, \tilde{\mathrm{A}}_t), \tilde{\varepsilon}_{t+1})\ \forall A_t, \varepsilon_{t+1}, \tilde{A}_t, \tilde{\varepsilon}_{t+1}$ with $\rho A_t + \Lambda\varepsilon_{t+1} = \rho\tilde{A}_t + \Lambda\tilde{\varepsilon}_{t+1}$. Thus $\omega_{t+1} = \widehat{\Xi}(K_{t+1}, \mathrm{A}_{t+1}) \equiv \widehat{\Psi}((0, K_{t+1}, \mathrm{A}_{t+1}/\rho), 0)$.

Accuracy test 1: $\hat{R}_t(K_t, A_t)$ is computed for 100 independent random vectors (K_t, A_t) at radius r from the steady state, for $r \in \{0.01, 0.10, 0.30\}$.[3] We report $T_r \equiv \max_{i,t} |\hat{R}_{i,t}|$.

Accuracy test 2: The model is simulated over 1000 periods (using (10)).[4] We compute $\hat{S}_t \equiv \max_i |\hat{R}_{i,t}|$ for $t \in \mathbf{T} \equiv \{10, 20, 30, .., 1000\}$ and report the maximum and the mean of $\hat{S}_t$ (across $\mathbf{T}$), denoted by $S_{\max}$ and S_{mean}, respectively.

Accuracy test 3: 200 simulation runs of 1000 periods are generated. Let $\hat{g}_{t+1} \equiv G(\hat{\omega}_{t+1}, \hat{\omega}_t, \varepsilon_{t+1})$, where $\{\hat{\omega}_t\}$ is the simulated series. For each run, we use a statistic described by den Haan and Marcet (1994, p.5) [DM] to test whether the errors $\hat{g}_{i,t+1}$ of the N countries' Euler equations (3) are orthogonal to a constant, and to first- and second order monomials of (K_t, A_t). Under the null hypothesis that the numerical solution is exact, the DM statistic has a χ^2 distribution. We compare the frequency distribution of the DM statistic (across the simulation runs) to the theoretical χ^2 distribution; DM argue that a close match between the two distributions indicates high solution accuracy. $P_{0.05}$ $[P_{0.50}]$ $\{P_{0.95}\}$ denote fractions of the simulated DM statistics below the 5% [50%] {95%} critical values of the χ^2 distribution.

5. Results

"Problem A" of DJJ consists of 8 models (A1,…,A8) defined by different functional forms for utility/production functions. Each of those models has to be solved for several values of N (number of countries). In total there are 30 different model variants.

5.1. Summary of results

Table 1 summarizes the results for the full set of 30 specifications, as well as for each of the 8 models. For each (sub)set of specifications, we report the maximum of T_r and of $S_{\max}$, and the average of S_{mean} across the individual specifications (included in that set).[5] Cols. 2-6 and Cols. 7-11 show results for a linear model solution and for the quadratic (second-order accurate) solution, respectively (the linear solution is obtained by just using the first-order terms of (10)). It seems interesting to compare these two solutions, as linear solutions have widely been used in macroeconomics. Below, $T_r^L, S_{\max}^L, S_{mean}^L$, $\{T_r^Q, S_{\max}^Q, S_{mean}^Q\}$ refer to the linear {quadratic} approximation.

The row labeled "ALL" reports maxima [averages] of $T_r, S_{\max}$ $[S_{mean}]$ across all 30 specifications. The row labeled "A1",..,"A8" shows maxima [averages] of $T_r, S_{\max}$ $[S_{mean}]$ across the individual specifications of model A1,..,A8.

For ***each*** of the 30 specifications, T_r (for *all* values of r considered here), $S_{\max}$ and S_{mean} are lower under the quadratic approximation than under the linear approximation. The

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[3] We generate the random K_t, A_t as follows: let h be a column vector with $2N$ i.i.d. elements, and $\tilde{h} \equiv rh/(h'\cdot h)^{.5}$ (NB $(\tilde{h}'\cdot\tilde{h})^{.5} = r$); we set $(K_t, A_t) = \log(1+\tilde{h})$ (steady state logged capital and productivity are zero).

[4] All simulations use the Kim, Kim, Schaumburg and Sims (2003) "pruning" approach (that drops terms involving 3rd and higher-order powers of the state variables from the recursion). The steady state is used for initial values; the actual length of each run was 1200 periods--the first 200 periods were discarded to ensure independence from initial conditions.

[5] Test 3 requires much longer computing times than tests 1-2, and was only computed for a few specifications (Table 3).

maxima of $S^Q_{\max}$ and $S^L_{\max}$ across *all* 30 variants are $10^{-2.71}$=0.19% and $10^{-1.78}$=1.65%, respectively, while the averages of S^Q_{mean} and S^L_{mean} across *all* variants are $10^{-3.72}$=0.019% and $10^{-2.50}$=0.31%, respectively.

Accuracy is highest when the system is close to the steady state: T_r is increasing in r (distance from steady state). Across all 30 specifications, the maxima of $T^Q_{0.01}$ and $T^L_{0.01}$ (r=0.01) are $10^{-4.45}$= 0.0035% and $10^{-2.66}$=0.22%, respectively; the maxima of $T^Q_{0.3}$ and $T^L_{0.3}$ (r=0.3) are $10^{-1.08}$= 8.31% and $10^{-0.59}$=25.70%, respectively. In all 30 cases, $T^L_{.3}-T^Q_{.3}>T^L_{.1}-T^Q_{.1}>T^L_{.01}-T^Q_{.01}>0$ holds. Hence, the accuracy gain from using the quadratic approximation (instead of the linear approximation) is larger when state variables are farther away from the steady state.

It seems interesting to investigate whether accuracy depends on the number of countries (N) or the curvature of preferences. To this end, we regressed the logs of $S^L_{\max}, S^L_{mean}, S^Q_{\max}, S^Q_{mean}$ and $S^L_{\max}-S^Q_{\max}, S^L_{mean}-S^Q_{mean}$ (for all 30 variants) on: a constant; the number of countries N and the household's intertemporal elasticity of substitution (inverse of the coefficient of relative risk aversion) γ. The results are shown in **Table 2**. Approximation errors and the accuracy gain produced by the quadratic approximation are increasing in the risk aversion coefficient; by contrast, errors are not systematically linked to *N*.

5.2. Detailed results for selected cases

Table 3 shows detailed results for 16 individual specifications: each of the eight models A1-A8 is solved for N=2 and N=6 countries, respectively.

Table 3 confirms that the error measures $T_r, S_{\max}, S_{mean}$ are not closely linked to the number of countries. In several cases, the accuracy gain produced by the quadratic approximation is substantial. The largest accuracy gain occurs for Problem A7 with N=2; there, $T^L_{.3}=10^{-0.59}$=25.70%, $S^L_{\max}=10^{-1.80}$=1.58%, $T^Q_{.3}=10^{-1.13}$=7.41%, $S^Q_{\max}=10^{-2.74}$= 0.18%; thus, in this case, $T_{.3}$ and $S_{\max}$ are lower by 18.3 and 1.4 percentage points, respectively, under the quadratic approximation (compared to the linear approximation).

The simulated DM frequencies ($P_{0.05}, P_{0.50}, P_{0.95}$) are similar across the quadratic and linear approximations—however, the simulated frequencies favor *very* slightly the quadratic approximation.[6] Under both approximations, the DM accuracy measures in Table 3 are *much* "worse" when the number of countries is large (N=6) than when N=2. E.g., for Problem A1, the quadratic approximation gives $P_{0.05}$=0.02, $P_{0.50}$=0.41, $P_{0.95}$=0.95 when N=2, compared to $P_{0.05}$=0.00, $P_{0.50}$=0.05, $P_{0.95}$=0.58 when N=6. The DM accuracy measure is sensitive to the number of instruments and to the length of the simulated series. For a sufficiently large number of instruments (and sufficiently long series), any approximate model solution fails the DM test (see discussion in DM, p.7). In Table 3, the number of instruments is $1+3N+2N^2$, i.e. there are 15 instruments when N=2, and 91 instruments when N=6. Thus the number of instruments is much larger when N=6 (the largest number of instruments used by DM (1994) was 7). In experiments with less instruments and/or shorter series we detected no dependence of the DM accuracy measure on the number of countries (results available on request).

[6] 28 of the 48 simulated $P_{0.05}, P_{0.50}, P_{0.95}$ frequencies are closer to 0.05, 0.5 or 0.95, respectively, under the quadratic approximation than under the linear approximation; only 7 of the simulated frequencies are *less* close under the quadratic approximation.

5.3. Computing times

Table 4 reports the time required to solve the model and run the accuracy tests, using MATLAB 6.1 on a Pentium 4 PC (2.5 GHz).[7] The Columns labeled "Deriv." and "Algor." show the time it takes to compute the derivatives of the model, and the time it takes to compute the solution with gensys2 (using the derivatives). For the quadratic approximation, the former is about 0.1 seconds when there are $N=2$ countries and about 35 seconds when $N=10$; gensys2 takes less than 0.1 second to compute the solution when $N=2$, and about 3 seconds when $N=10$. Even when the number of countries is large, gensys2 is thus very fast.

[7] The inversion of a 1000×1000 matrix with i.i.d. random elements takes about 1.9 seconds on that machine.

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Table 1. Accuracy tests: summary

Mo= del	Linear approximation: Test 1 $T_{.01}$	$T_{.1}$	$T_{.3}$	Linear approximation: Test 2 $S_{\max}$	S_{mean}	2nd order approximation: Test 1 $T_{.01}$	$T_{.1}$	$T_{.3}$	2nd order approximation: Test 2 $S_{\max}$	S_{mean}
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
ALL	-2.66	-1.40	-0.59	-1.78	-2.50	-4.45	-2.42	-1.08	-2.71	-3.72
A1	-3.12	-1.84	-0.99	-2.25	-2.84	-5.37	-3.04	-1.68	-3.44	-4.24
A2	-3.07	-1.80	-0.96	-2.19	-2.74	-5.25	-2.93	-1.57	-3.21	-3.94
A3	-2.73	-1.57	-0.74	-1.81	-2.28	-4.82	-2.51	-1.16	-2.82	-3.45
A4	-2.78	-1.60	-0.77	-1.89	-2.35	-4.87	-2.55	-1.21	-2.85	-3.53
A5	-3.12	-1.83	-0.98	-2.26	-2.81	-5.35	-3.02	-1.67	-3.41	-4.19
A6	-2.85	-1.68	-0.82	-1.85	-2.52	-4.82	-2.65	-1.29	-2.72	-3.78
A7	-2.66	-1.40	-0.59	-1.78	-2.23	-4.45	-2.42	-1.08	-2.71	-3.40
A8	-2.76	-1.57	-0.75	-1.81	-2.32	-4.79	-2.52	-1.18	-2.82	-3.52

Note: Row 1 (labeled 'ALL') summarizes the results for all 30 model specifications; the remaining rows summarize the results for all specifications of model A1 (row labeled A1), model A2 etc. For each (sub)set of model specifications, the Cols. labeled "T_r" $(r \in \{.01, .1, .3\})$ and "$S_{\max}$" (i.e. Cols. (2)-(5); (7)-(10)) report maxima of error measures $T_r, S_{\max}$ across the individual model specifications; the Cols. labeled "S_{mean}" (i.e. Cols. (6) and (11)) show averages of S_{mean} across individual model specifications. Cols. (2)-(6) [Cols. (7)-(11)]: linear [quadratic] model solution.

The reported Figures are logarithms to the base 10 $(\log_{10})$ of the maxima/averages of $T_r, S_{\max}, S_{mean}$.

Table 2. Relating accuracy to model parameters: regression results

Dependent variable		Constant	N^i	γ^i	R^2
$\log_{10}(S^{L,i}_{\max})$	=	-1.82	-0.009 N^i	-0.37 γ^i	R^2= .26
		(19.69)	*(0.65)*	*(2.86)*	
$\log_{10}(S^{Q,i}_{\max})$	=	-2.84	+0.006 N^i	-0.66 γ^i	R^2= .28
		(20.41)	*(0.26)*	*(3.36)*	
$\log_{10}(S^{L,i}_{mean})$	=	-2.35	+0.009 N^i	-0.50 γ^i	R^2= .27
		(21.68)	*(0.54)*	*(3.31)*	
$\log_{10}(S^{Q,i}_{mean})$	=	-3.46	+0.008 N^i	-0.78 γ^i	R^2= .37
		(25.84)	*(3.91)*	*(4.12)*	
$\log_{10}(S^{L,i}_{\max} - S^{Q,i}_{\max})$	=	-1.86	-0.011 N^i	-0.35 γ^i	R^2= .26
		(21.01)	*(0.76)*	*(2.81)*	
$\log_{10}(S^{L,i}_{mean} - S^{Q,i}_{mean})$	=	-2.38	+0.009 N^i	-0.49 γ^i	R^2= .27.
		(22.27)	*(0.55)*	*(3.26)*	

Note: Regressions of logged accuracy measures for the 30 model variants on a constant and on parameters used in these variants are shown. $S^{L,i}_{\max}$, $S^{Q,i}_{\max}$, $S^{L,i}_{mean}$, $S^{Q,i}_{mean}$: accuracy measures for model variant i; N^i, γ^i: number of countries and intertemporal substitution elasticity in for variant *i*. (For model variants in which γ differs across countries, the regression uses the mean of γ across countries.) $S^{L,i}_{\max}$ and $S^{L,i}_{mean}$ [$S^{Q,i}_{\max}$ and $S^{Q,i}_{mean}$] pertain to the linear [quadratic] approximation.

The figures in parentheses below the regression coefficients are absolute values of t-statistics.

Table 3. Accuracy tests: results for selected specifications

Mo=		Linear approximation							2nd order approximation						
		Test 1		Test 2		Test 3			Test 1		Test 2		Test 3		
del	N	$T_{.01}$	$T_{.3}$	$S_{\max}$	S_{mean}	$P_{.05}$	$P_{.5}$	$P_{.95}$	$T_{.01}$	$T_{.3}$	$S_{\max}$	S_{mean}	$P_{.05}$	$P_{.5}$	$P_{.95}$
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*	*(16)*
A1	2	-3.26	-0.99	-2.34	-3.02	.03	.41	.96	-5.40	-1.71	-3.66	-4.44	.02	.41	.95
A1	6	-3.15	-1.51	-2.35	-2.80	.00	.05	.57	-5.41	-2.19	-3.69	-4.23	.00	.05	.58
A2	2	-3.20	-0.94	-2.25	-2.82	.01	.40	.95	-5.37	-1.57	-3.40	-4.07	.02	.42	.94
A2	6	-3.08	-1.45	-2.24	-2.71	.00	.04	.59	-5.28	-2.07	-3.28	-3.91	.00	.05	.59
A3	2	-2.88	-0.74	-1.88	-2.45	.01	.38	.94	-4.95	-1.32	-2.93	-3.64	.01	.42	.95
A3	6	-2.73	-1.14	-1.81	-2.18	.00	.03	.49	-4.82	-1.79	-2.85	-3.35	.00	.04	.52
A4	2	-2.92	-0.77	-1.94	-2.52	.01	.40	.95	-4.99	-1.36	-3.06	-3.71	.01	.43	.95
A4	6	-2.78	-1.19	-1.89	-2.25	.00	.05	.56	-4.87	-1.81	-2.91	-3.45	.00	.05	.57
A5	2	-3.26	-0.98	-2.32	-2.92	.01	.41	.95	-5.43	-1.68	-3.71	-4.36	.01	.44	.95
A5	6	-3.15	-1.50	-2.33	-2.79	.00	.05	.58	-5.40	-2.17	-3.57	-4.18	.00	.05	.59
A6	2	-3.03	-0.83	-1.98	-2.69	.01	.40	.95	-4.88	-1.44	-3.16	-3.96	.00	.43	.95
A6	6	-2.89	-1.28	-2.00	-2.47	.00	.04	.53	-4.85	-1.92	-2.98	-3.77	.00	.04	.55
A7	2	-2.80	-0.59	-1.80	-2.37	.02	.38	.92	-4.45	-1.13	-2.74	-3.55	.01	.41	.95
A7	6	-2.66	-1.07	-1.81	-2.15	.00	.03	.47	-4.55	-1.70	-2.71	-3.30	.00	.04	.49
A8	2	-2.91	-0.75	-1.95	-2.47	.01	.40	.94	-4.79	-1.31	-3.01	-3.72	.01	.42	.95
A8	6	-2.76	-1.16	-1.83	-2.23	.00	.03	.50	-4.79	-1.82	-2.82	-3.41	.00	.04	.53

Note: Cols. 1-2 list the model variant and the number of countries (N). The figures shown for Tests 1 and 2 are logs of error measures $(\log_{10}(T_r), \log_{10}(S_{\max}), \log_{10}(S_{mean}))$.

Table 4. Computing times (seconds)

	Linear approximation					2nd order approximation				
N	**Deriv.**	**Algor.**	**Test 1**	**Test 2**	**Test 3**	**Deriv.**	**Algor.**	**Test 1**	**Test 2**	**Test 3**
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
2	0.01	0.06	12	12	161	0.09	0.09	13	15	364
10	0.03	0.39	56	50	1838	35.95	3.36	89	96	3586

Note: Columns labeled **"Deriv.", "Algor."**: computing time of derivatives of model, and computing time of the solution, respectively. Cols. Labeled **"Test 1":** computing time for accuracy statistic T_r. Cols. labeled **"Test 2" ["Test 3"]**: computing time for accuracy statistics S_{max}, S_{mean} $[P_{.05}, P_{.5}, P_{.95}]$. ***N***: number of countries.