Work in progress

Solving Non-Linear Rational Expectations Models: Approximations based on Taylor Expansions

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This paper presents an algorithm that computes Taylor series expansions of order k=2, 3, 4 of the policy function of non-linear rational expectations models. Approximations of order $k \geq 2$ may be markedly more accurate than linear approximations (k=1) *if* the variability of the exogenous shocks is high and/or the model exhibits strong curvature.

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1. Introduction

Many widely studied stochastic rational expectations models can be expressed as:

$$\underset{n\times 1}{E_t G}(\underset{n\times 1}{\omega_{t+1}}, \omega_t, \underset{1\times 1}{\xi}\cdot\underset{m\times 1}{\varepsilon_{t+1}}) = \underset{n\times 1}{0}, \qquad t \geq 0 \tag{1}$$

where E_t denotes the mathematical expectation conditional upon complete information about periods t and earlier; $G: R^{2n+m} \to R^n$ is a function, and ω_t is an $n\times 1$ vector of variables known at date t; $\xi \geq 0$ is a scalar, and $\varepsilon_{t+1} = (\varepsilon_{1,t+1}; \varepsilon_{2,t+1}; ...; \varepsilon_{m,t+1})$ is an $m\times 1$ vector of date t+1 exogenous independent random variables. The following discussion assumes that ε_t has bounded support and these moments: $E_t\varepsilon_{i,t+1} = 0$, $E_t(\varepsilon_{i,t+1})^2 = 1$, $E_t(\varepsilon_{i,t+1})^3 = 0$, $E_t(\varepsilon_{i,t+1})^4 = 3$. [1]

The solution of (1) is a "policy function"

$$\omega_{t+1} = f(\omega_t, \xi\varepsilon_{t+1}, \xi), \tag{2}$$

that satisfies the condition

$$E_t G(f(\omega_t, \xi\varepsilon_{t+1}, \xi), \omega_t, \xi\varepsilon_{t+1}) = 0 \quad \forall\omega_t, \forall\xi \geq 0. \tag{3}$$

When G is linear, then the solution f is likewise linear and can easily be computed using well-known algorithms (e.g., Hansen and Sargent (1980), Blanchard and Kahn (1980), Anderson and Moore (1985), Klein (2000) and Sims (2002)). However, most economic models (G) are non-linear. A widely used approach (e.g., King, Plosser and Rebelo, 1988) consists in taking a linear approximation of non-linear models, around a deterministic steady state. A drawback of that approach is that it does not allow to capture the effect of the volatility of exogenous shocks on the *mean* values of endogenous variables, as the linearized solution exhibits certainty-equivalence; that method is thus not suited for computing welfare or for the analysis of risk premia on financial assets.

Judd and Guu (1993), Judd and Gaspar (1996) and Judd (1998) propose a general approach for approximating the policy functions of continuous time and discrete time models using Taylor expansions of order k>1, around a steady state. To date, applications of that approach to discrete time models have mainly focused on quadratic approximations, k=2. See, e.g., Sims (2000), Collard and Juillard (2001), Schmitt-Grohé and Uribe (2004), and Schaumburg (2002) who have produced (and made publicly available) computer programmes for k=2; several studies have used these programmes for the analysis of medium scale macroeconomic models (e.g., Kollmann (2002, 2003, 2004), Schmitt-Grohé and Uribe (2003) and Kim (2003)). Judd and Jin (2002), Jin (2003) and Juillard (2003) recently developed computer code for discrete time approximations of order k>2 (I learnt about these contributions after completing most of the work described here).

This paper presents an algorithm for computing approximations of order k=2, 3 and 4, using an approach that differs from that used by the papers that were just cited (see discussion in Sect. 2.3 below). The computational approach used here differs from that used by the papers cited above. MATLAB code that implements the present algorithm will be made available on my web page.

[1] The first to fourth moments of $\varepsilon_{i,t+1}$ correspond thus to those of a standard normal random variable. Note that $E_t(\varepsilon_{i,t+1})^3 = 0$ holds for any symmetric distribution. The algorithm can easily be adapted to allow for $E_t(\varepsilon_{i,t+1})^4 \neq 3$.

It appears that approximations of order $k \geq 2$ may be markedly more accurate than linear approximations (k=1). A fourth order approximation may be noticeably more accurate than a second order approximation *if* the variability of the exogenous shocks is high and/or the model exhibits strong curvature.

Section 2 describes the algorithm (and compares it to previous methods). Section 3 applies it to selected models.

2. The method

I begin by discussing definitions/notation. Throughout this paper, the term "steady state" refers to the deterministic steady state, i.e. to a model solution in which $\omega_{t+1} = \omega_t = \omega$, $\varepsilon_{t+1} = 0_m \quad \forall t$, with $G(\omega, \omega, 0_m) = 0_n$, where 0_m is a column vector of zeros (m elements). Steady state values are denoted by variables without time subscripts, and $dz_t = z_t - z$ is the deviation of a variable z_t from its steady state value.

R_n denotes a polynomial consisting of powers of order n and higher of elements of $\{d\omega_\tau ; \xi\varepsilon_\tau\}_{\tau \geq 0}$.

$h_t^{(n)}$ denotes an *n-th* order accurate approximation of variable h_t, in the following sense: $h_t - h_t^{(n)} = R_{n+1}$. Let $h_t^{\{s\}} = h_t^{(s)} - h_t^{(s-1)}$, for $s > 1$. Thus, $h_t^{(2)} = h_t^{(1)} + h_t^{\{2\}}$, $h_t^{(3)} = h_t^{(1)} + h_t^{\{2\}} + h_t^{\{3\}}$ etc.

If a and b are matrices, then $(a;b)$ denotes the matrix obtained by vertically concatenating a and b (provided a and b have the same number of columns), while (a,b) denotes horizontal concatenation.

Let k be a column vector with N elements. $P_n(k)$, for $n = 2,3,...$ denotes a column vector consisting of all $n-th$ order powers and cross-products of the elements of k. In the computer programs, these powers/cross-products are arranged in the following order: $P_1(k) = k$ and $P_{s+1}(k) = (k_1 P_s(k); k_2 P_s((k_2; k_3; ..; k_N)); ....; k_{N-1} P_s((k_{N-1}; k_N)); k_N P_s(k_N))$ for $s > 1$, where k_i is the $i-th$ element of k.

2.1. First-order approximations

The algorithm for generating second (and higher) order approximations presented here takes as its starting point a first-order accurate (linear) model solution. As discussed above, several solution methods for linear(ized) rational expectations models are available in the literature. Any of these methods could be used to generate higher order accurate solutions. Here, I use Sims' (2002) algorithm (that can be implemented using Chris Sims' computer program gensys, available at *www.princeton.edu/~sims*). This section briefly reviews Sims' (2002) approach. Following Sims (2002) note that (1) implies

$$G(\omega_{t+1}, \omega_t, \xi\varepsilon_{t+1}) + \Pi\eta_{t+1} = 0, \text{ with } E_t\eta_{t+1} = 0, \qquad (2)$$

where Π is a matrix of size $n \times p$, where p equals the number of model equations that include date t expectations of date t+1 variables. η_t is function of ε_{t+1} (that function is not known a priori).

Sims (2002) shows that the solution of the model can be written as:

$$y_{t+1} = F(y_t, \xi\varepsilon_{t+1}), \quad x_{t+1} = M(y_{t+1}),$$
$$\text{with } (y_t; x_t) = Z\,\omega_t, \qquad (3)$$

where Z is a non-singular $n \times n$ matrix. y_t and x_t are column vectors with n_y and n_x elements, respectively, with $n = n_y + n_x$; F and M are functions.

Take a first-order Taylor expansion of (2) around a steady state. This gives:

$$G0\, d\omega_{t+1} = G1\, d\omega_t + G2\xi\varepsilon_{t+1} + \Pi\,\eta_{t+1} + R_2\,, \tag{4}$$

where $G0, G1$ and $G2$ are matrices/vectors of size $n\times n$, $n\times n$ and $n\times 1$, respectively.

Using (3), we can write (4) as:

$$K0\ (dy_{t+1}; x_{t+1}) = K1\ (dy_{t+1}; x_{t+1}) + G2\xi\varepsilon_{t+1} + \Pi\,\eta_{t+1} + R_2\,, \tag{5}$$

with $K0 = G0\, Z^{-1}$, $K1 = G1\, Z^{-1}$.

Sims (2002) shows there exists an $n\times n$ matrix T with the following properties: [2]

$$T\ K0 = \begin{bmatrix} I_{n_y} & H1 \\ 0 & J1 \end{bmatrix},\ \ T\ K1 = \begin{bmatrix} F1 & H2 \\ 0 & J2 \end{bmatrix} \text{ and } T\ \Pi = \begin{bmatrix} 0_{n_y\times p} \\ \Pi^* \end{bmatrix}, \tag{6}$$

where $H1\ (n_y\times n_x)$, $J1\ (n_x\times n_x)$, $F1\ (n_y\times n_y)$, $H2\ (n_y\times n_x)$, $J2\ (n_x\times n_x)$, $\Pi^*\ (n_x\times p)$ are matrices (sizes shown in parentheses). Premultiplying (5) by T thus gives a block-recursive system of equations:

$$dy_{t+1} + H1 dx_{t+1} = F1 dy_t + H2 dx_t + F2\,\xi\varepsilon_{t+1} + R_2\,, \tag{7}$$

$$J1\, dx_{t+1} = J2\ dx_t + G22\ \xi\varepsilon_{t+1} + \Pi^*\eta_{t+1} + R_2\,, \tag{8}$$

where $F2$ and $G22$ are matrices with n_y and n_x rows respectively ($(F2; G22)\ =\ T\ G2$).

The assumed stationarity of the model solution implies that the eigenvalues of $F1$ and $(J2)^{-1}J1$ are inside the unit circle. Solving forward (8) yields:

$$dx_t = -\{\sum_{j=0}^{j=\infty} ((J2)^{-1}J1)^j\ (J2)^{-1}(G22\ \xi\varepsilon_{t+1+j} + \Pi^*\ \eta_{t+1+j})\}\ \ +\ \ R_2\,. \tag{9}$$

As $dx_t = E_t dx_t$ holds, and $E_t\varepsilon_{t+j+s} = E_t\eta_{t+j+s} = 0\ \ \forall\, s\geq 0$, (9) implies: $dx_t = R_2$, and thus:

$$dx_t^{(1)} = 0\,. \tag{10}$$

(9) and (7) imply:

$$dy_{t+1}^{(1)} = F1\ dy_t + F3\ \xi\varepsilon_{t+1}\,. \tag{11}$$

2.2. Higher order approximations

To $n-th$ order accurate solutions ($n\geq 2$), take an $n-th$ order Taylor expansion of (2):

$$G0\, d\omega_{t+1} = G1\, d\omega_t + G2\xi\varepsilon_{t+1} + \sum\nolimits_{i=2}^{i=n}\Theta i\, P_i(d\Psi_{t+1}) + \Pi\,\eta_{t+1} + R_{n+1}\,, \tag{12}$$

where $\Psi_{t+1} = (\omega_{t+1}; \omega_t; \xi\varepsilon_{t+1})$, while $\Theta 1,\ldots,\Theta n$ are matrices. Using (3), we can transform (12) into:

$$K0\ (dy_{t+1}; x_{t+1}) = K1\ (dy_t; x_t) + G2\xi\varepsilon_{t+1} + \sum\nolimits_{i=2}^{i=n}\Omega i\, P_i(d\Lambda_{t+1}) + \Pi\,\eta_{t+1} + R_{n+1}\,, \tag{13}$$

with $\Lambda_{t+1} = (y_{t+1}; x_{t+1}; y_t; x_t; \xi\varepsilon_{t+1})$, where $\Omega 1,\ldots,\Omega n$ are matrices. [3]

Premultiplying (13) by T (see (6)) yields:

$$dy_{t+1} + H1\ dx_{t+1} = F1\ dy_t + H2\ dx_t + F2\ \xi\varepsilon_{t+1} + \sum\nolimits_{i=2}^{i=n}\Phi i1\, P_i(d\Lambda_{t+1}) + R_{n+1}\,, \tag{14}$$

$$J1\ dx_{t+1} = J2\ dx_t + G22\xi\varepsilon_{t+1} + \Pi^*\eta_{t+1} + \sum\nolimits_{i=2}^{i=n}\Phi i2\, P_i(d\Lambda_{t+1}) + R_{n+1}\,, \tag{15}$$

where $\Phi i1$ and $\Phi i2$ are matrices: $(\Phi i1; \Phi i2) = T\ \Omega i$ for $i = 2,\ldots,n$.

Solving (15) forward and taking conditional expectations gives:

[2] In Chris Sims' gensys program, T corresponds to the product of the matrices *tmat* and q: $T\ =\ tmat * q$.

[3] Ωi (for $i = 1,\ldots,n$) is a function of Θi and of Z. Determining Ωi is simple but tedious. Interested readers may consult the computer code for the details. Analogous remarks apply to many of the coefficients in the rest of this paper.

$$dx_t^{(n)} = -\sum_{s=0}^{s=\infty}((J2)^{-1}J1)^s \ (J2)^{-1}E_t\{\sum_{i=2}^{i=n}\Phi i2(P_i(d\Lambda_{t+1+s}))^{(n)}\}. \tag{16}$$

Thus, to determine $dx_t^{(n)}$ we have to compute the path of the $2nd$ to $n-th$ order powers and cross-products of the state variables. Given such a path, and a solution for $dx_t^{(n)}$, the time path for $(dy_{t+1})^{(n)}$ can be determined recursively using (14). An $n-th$ order accurate solution for ω_t can then be computed using (3): $(d\omega_t)^{(n)} = Z^{-1}((dy_t)^{(n)};(dx_t)^{(n)})$.

The algorithm described below is based on the fact that that $P_i(d\Lambda_{t+1})^{(n)}$ (for $i=2,...,n$) can be determined from $(d\Lambda_{t+1})^{(1)}, (d\Lambda_{t+1})^{(2)}, ., (d\Lambda_{t+1})^{(n-1)}$. For example, a ***second order*** accurate model solution requires knowledge of $P_2(d\Lambda_t)^{(2)}$. $P_2(d\Lambda_t)$ is a vector consisting of the products of the elements of $d\Lambda_t$. Let dk_t and dq_t be two elements of $d\Lambda_t$. Note that

$$(dk_t \ dq_t)^{(2)} = dk_t^{(1)} \ dq_t^{(1)}. \text{[4]} \tag{17}$$

To generate a ***third order*** accurate model solution, we need a third order accurate evaluation of products of pairs and triplets of elements of the vector $d\Lambda_t$. Such an evaluation can be obtained from first- and second order accurate model solution, as the product of two variables dk_t and dq_t can be expressed as:

$$(dk_t \ dq_t)^{(3)} = (dk_t)^{(1)}(dq_t)^{(1)} + (dk_t)^{(1)}(dq_t)^{\{2\}} + (dk_t)^{\{2\}}(dq_t)^{(1)}, \tag{18}$$

while a *third* order accurate approximation of the product of three variables dk_t, dq_t, dr_t is given by:

$$(dk_t \ dq_t \ dr_t)^{(3)} = (dk_t)^{(1)} \ (dq_t)^{(1)} \ (dr_t)^{(1)}. \tag{19}$$

A **fourth order** accurate model solution requires a fourth order accurate evaluation of products of pairs, triplets, and quadruplets of elements of the vector $d\Lambda_t$. This can be obtained from first order, second order and third order accurate solutions. Note that

$$(dk_t dq_t)^{(4)} = (dk_t)^{(1)}(dq_t)^{(1)} + (dk_t)^{(1)}(dq_t)^{\{2\}} + (dk_t)^{\{2\}}(dq)^{(1)} + (dk_t)^{(1)}(dq_t)^{\{3\}} + (dk_t)^{\{3\}}(dq)^{(1)} + (dk_t)^{\{2\}}(dq_t)^{\{2\}}, \tag{20}$$

$$(dk_t \ dq_t \ dr_t)^{(4)} = (dk_t)^{(1)}(dq_t)^{(1)}(dr_t)^{(1)} + (dk_t)^{\{2\}}(dq_t)^{(1)}(dr_t)^{(1)} + (dk_t)^{(1)}(dq_t)^{\{2\}}(dr_t)^{(1)} + (dk_t)^{(1)}(dq_t)^{(1)}(dr_t)^{\{2\}}, \tag{21}$$

$$(dk_t \ dq_t \ dr_t \ ds_t)^{(4)} = (dk_t)^{(1)}(dq_t)^{(1)}(dr_t)^{(1)}(ds_t)^{(1)}. \tag{22}$$

2.2.1. Second order accurate solution

For $n=2$, (14) and (16) are given by:

$$dy_{t+1} + H1 \ dx_{t+1} = F1 \ dy_t + H2 \ dx_t + F2 \ \xi\varepsilon_{t+1} + \Phi21 \, P_2(d\Lambda_{t+1}) + R_3, \tag{23}$$

$$dx_t^{(2)} = -\sum_{s=0}^{s=\infty}((J2)^{-1}J1)^s \ (J2)^{-1}E_t\Phi22(P_2(d\Lambda_{t+1+s}))^{(2)}. \tag{24}$$

Let

$$Z2_{t+1} \equiv (\xi^2; P_2((dy_t;\xi\varepsilon_{t+1}))), \tag{25}$$

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[4] Note that $dk_t dq_t = (dk_t^{(1)} + R_2)(dq_t^{(1)} + R_2) = dk_t^{(1)} dq_t^{(1)} + R_2$.

$$h2_t \equiv (\xi^2; P_2(dy_t)). \tag{26}$$

(17) implies that $(P_2(d\Lambda_{t+1}))^{(2)} = P_2((d\Lambda_{t+1})^{(1)})$. It follows from (10),(11) that

$$(d\Lambda_{t+1})^{(1)} = (F1 dy_t + F2\xi\varepsilon_{t+1}; 0_{n_x}; dy_t; 0_{n_x}; \xi\varepsilon_{t+1}). \tag{27}$$

Hence, $(P_2(d\Lambda_{t+1}))^{(2)}$ is a linear function of the squares and cross-products of the vector $(dy_t;\xi\varepsilon_{t+1})$, and thus of the elements of $P_2((dy_t;\xi\varepsilon_{t+1}))$. Thus we can write

$$\Phi21\,(P_2(d\Lambda_{t+1}))^{(2)} = \Psi21\ Z2_{t+1}, \qquad \Phi22\,(P_2(d\Lambda_{t+1}))^{(2)} = \Psi22\ Z2_{t+1}, \tag{28}$$

for some matrices $\Psi21$, $\Psi22$.

As dy_t and ε_{t+1} are independent, $E_t Z2_{t+1}$ is a linear function of ξ^2 and of $P_2(dy_t)$, and thus:

$$E_t Z2_{t+1} = \Xi2\ h2_t, \tag{29}$$

for some matrix $\Xi2$.

The logic that underlies (28) also implies that

$$(h2_{t+1})^{(2)} = \Psi23\ Z2_{t+1}, \tag{30}$$

for some matrix $\Psi23$.

Hence,

$$E_t(h2_{t+1})^{(2)} = \Psi23\ \Xi2\ h2_t \tag{31}$$

(28)-(31) imply that $\Phi22 E_t(P_2(d\Lambda_{t+1+s}))^{(2)} = \Psi22\ \Xi2\ (\Psi23\ \Xi2)^s\, h2_t$. Substituting this into (24) gives:

$$dx_t^{(2)} = S2\ h2_t, \text{ with } S2 \equiv -\sum_{s=0}^{s=\infty}((J2)^{-1}J1)^s\ (J2)^{-1}\ \Psi22\ \ \Xi2\ \ (\Psi23\,\Xi2)^s. \tag{32}$$

Using (28), (30) and (32), we can write (23) as:

$$(dy_{t+1})^{(2)} = F1\ dy_t + F2\ \xi\varepsilon_{t+1} + P2\ Z2_{t+1}, \tag{33}$$

for some matrix $P2$.

(32), (33) can also be expressed as:

$$dx_t^{(2)} = Q1\,\xi^2 + Q2\,P_2(dy_t), \tag{34}$$

$$(dy_{t+1})^{(2)} = F1\ dy_t + F2\ \xi\varepsilon_{t+1} + F3\,\xi^2 + F4\,P_2((dy_t;\xi\varepsilon_{t+1})), \tag{35}$$

where $Q1, Q2, F3$ and $F4$ are matrices/vectors.

(34),(35) have the same form as the second order accurate solutions derived by Sims (2000) and by Schmitt-Grohé and Uribe (2004). Application of the second-order accurate algorithm presented here to several models yielded coefficients $F1, F2, F3, F4$ and $M1, M2$ that are numerically indistinguishable from coefficients implied by the Sims (2000) algorithm.

2.2.2. Third order accurate solution

For n=3, (14) and (16) are given by:

$$dy_{t+1} + H1\ dx_{t+1} = F1\ dy_t + H2\ dx_t + F2\ \xi\varepsilon_{t+1} + \Phi21\,P_2(d\Lambda_{t+1}) + \Phi31\,P_3(d\Lambda_{t+1}) + R_4, \tag{36}$$

$$dx_t^{(3)} = -\sum_{s=0}^{s=\infty}((J2)^{-1}J1)^s\ (J2)^{-1}\ E_t\{\Phi22\,(P_2(d\Lambda_{t+1+s}))^{(3)} + \Phi32\,(P_3(d\Lambda_{t+1+s}))^{(3)}\}. \tag{37}$$

Let

$$Z3_{t+1} \equiv \left(\xi^2; P_2(dy_t;\xi\varepsilon_{t+1});\ \xi^2(dy_t;\xi\varepsilon_{t+1});\ P_3((dy_t;\xi\varepsilon_{t+1}))\right), \tag{38}$$

$$\text{and} \quad h3_t \equiv \left(\xi^2;\ P_2(dy_t);\ \xi^2 dy_t;\ P_3(dy_t)\right). \tag{39}$$

(30), (32), (34), (35) imply:

$$(d\Lambda_{t+1})^{\{2\}} = (F3\,\xi^2 + F4\,P_2((dy_t;\xi\varepsilon_{t+1});\, S2\,\Psi 23\,(\xi^2; P_2((dy_t;\xi\varepsilon_{t+1}));\, 0_{n_y};\, Q1\xi^2 + Q2\,P_2(y_t);\, 0_m), \quad (40)$$

i.e. $(d\Lambda_{t+1})^{\{2\}}$ is a linear function of $Z2_{t+1}$.

(17), (18), (19), (27), (40) imply that $(P_2(d\Lambda_{t+1}))^{(3)}$ can be expressed as linear functions of $Z3_{t+1}$. Thus:

$$\Phi 21\; P_2(d\Lambda_{t+1})^{(3)} + \Phi 31\; P_3(d\Lambda_{t+1})^{(3)} = \Psi 31\;\; Z3_{t+1}, \quad (41)$$

$$\Phi 22\; P_2(d\Lambda_{t+1})^{(3)} + \Phi 32\; P_3(d\Lambda_{t+1})^{(3)} = \Psi 32\;\; Z3_{t+1}, \quad (42)$$

for some matrices $\Psi 31$ and $\Psi 32$.

As dy_t and ε_{t+1} are independent, and as the third moments of the elements of the vector ε_{t+1} are zero, $E_t Z3_{t+1}$ is a linear function of $h3_t$:

$$E_t Z3_{t+1} = \Xi 3\;\; h3_t, \quad (43)$$

for some matrix $\Xi 3$.

The logic that underlies (41),(42) also implies that

$$(h3_{t+1})^{(3)} = \Psi 33\; Z3_{t+1}, \quad (44)$$

for some matrix $\Psi 33$. Hence,

$$E_t\,(h3_{t+1})^{(3)} = \Psi 33\; \Xi 3\;\; h3_t. \quad (45)$$

(42), (43), (45) imply $E_t\{\Phi 22\, P_2(d\Lambda_{t+1+s})^{(3)} + \Phi 32\; P_3(d\Lambda_{t+1+s})^{(3)}\} = \Psi 32\,\Xi 3\,(\Psi 33\;\Xi 3)^s\, h3_t$. Therefore,

$$dx_t^{(3)} = S3\; h3_t, \text{ with } S3 \equiv -\sum_{s=0}^{s=\infty}((J2)^{-1}J1)^s\;\; (J2)^{-1}\;\Psi 32\;\;\Xi 3\;\;(\Psi 33\;\Xi 3)^s. \quad (46)$$

Using (41), (44) and (46), we can write (36) as:

$$(dy_{t+1})^{(3)} = F1\; dy_t + F2\;\; \xi\varepsilon_{t+1} + P3\; Z3_{t+1}, \quad (47)$$

for some matrix P3.

(46), (47) imply:

$$dx_t^{(3)} = Q1\,\xi^2 + Q2\,P_2(dy_t) + Q3\;\xi^2\;\; dy_t + Q4\;P_3(dy_t), \quad (48)$$

$$(dy_{t+1})^{(3)} = F1\; dy_t + F2\;\; \xi\varepsilon_{t+1} + F3\,\xi^2 + F4\,P_2((dy_t;\xi\varepsilon_{t+1}) + F5\,\xi^2\;(dy_t;\xi\varepsilon_{t+1}) + F6\,P_3((dy_t;\xi\varepsilon_{t+1}), \quad (49)$$

where $Q1, Q2, Q3, Q4, F3, F4, F5$ and $F6$ are matrices/vectors.

The coefficients of the first order terms in (49) (i.e. $F1, F2$) are, by construction identical to the corresponding coefficients in the first- and second order accurate solutions (11), (35). It appears that the coefficients of the second order terms $Q1, Q2, F3, F4$ are also identical across the second- and third order accurate solutions (34)-(35) and (48)-(49). A proof of this is provided in the Appendix.

2.2.3. Fourth order accurate solution

The derivation of a fourth order accurate solution follows the same logic as the previous discussions. For n=4, (14) and (16) are given by:

$$dy_{t+1} + H1\; dx_{t+1} = F1\,dy_t + H2\,dx_t + F2\;\xi\varepsilon_{t+1} + \Phi 21\,P_2(d\Lambda_{t+1}) + \Phi 31\,P_3(d\Lambda_{t+1}) + \Phi 41\,P_4(d\Lambda_{t+1}) + R_5, \quad (50)$$

$$dx_t^{(4)} = -\sum_{s=0}^{s=\infty}((J2)^{-1}J1)^s (J2)^{-1} E_t\{\Phi 22(P_2(d\Lambda_{t+1+s}))^{(4)} + \Phi 32(P_3(d\Lambda_{t+1+s}))^{(4)} + \Phi 42(P_4(d\Lambda_{t+1+s}))^{(4)}\}. \quad (51)$$

(44), (46), (47) and (48), imply:

$$(d\Lambda_{t+1})^{\{3\}} = (F5\,\xi^2\;(dy_t;\xi\varepsilon_{t+1}) + F6\,P_3((dy_t;\xi\varepsilon_{t+1});\, S3\,\Psi 33\, Z3_{t+1};$$

$$0_{n_y};\ Q3\ \xi^2\ dy_t + Q4\ P_3(dy_t);\ 0_m). \quad (52)$$

Let

$$Z4_{t+1} \equiv (\xi^2;\ P_2(dy_t;\xi\varepsilon_{t+1});\ \xi^2(dy_t;\xi\varepsilon_{t+1});\ P_3((dy_t;\xi\varepsilon_{t+1});\xi^4;\xi^2 P_2((y_t;\varepsilon_{t+1}));P_4((y_t;\varepsilon_{t+1})), \quad (50)$$

$$h4_t \equiv (\xi^2;\ P_2(dy_t);\ \xi^2 dy_t;\ P_3(dy_t);\ \xi^4;\ \xi^2 P_2(dy_t);\ P_4(dy_t)). \quad (51)$$

Using (17)-(22), (27), (40) and (43), we obtain:

$$\Phi 21\,(P_2(d\Lambda_{t+1}))^{(4)} + \Phi 31\,(P_3(d\Lambda_{t+1}))^{(4)} + \Phi 41\,(P_4(d\Lambda_{t+1}))^{(4)} = \Psi 41\ Z4_{t+1},$$

$$\Phi 22\,(P_2(d\Lambda_{t+1}))^{(4)} + \Phi 32\,(P_3(d\Lambda_{t+1}))^{(4)} + \Phi 42(P_4(d\Lambda_{t+1}))^{(4)} = \Psi 42\ Z4_{t+1},$$

for some matrices $\Psi 41$ and $\Psi 42$. Also,

$$E_t Z4_{t+1} = \Xi 4\, h4_t,$$

and $\quad (h4_{t+1})^{(4)} = \Psi 43\ Z4_t,\ \ E_t(h4_{t+1})^{(4)} = \Psi 43\ \Xi 4\ h4_t,$

for some matrices $\Xi 4$ and $\Psi 43$. We have

$$(dx_t)^{(4)} = S4\ h4_t, \text{ with } S4 \equiv -\sum_{s=0}^{s=\infty}((J2)^{-1}J1)^s\ (J2)^{-1}\ \Psi 32\ \ \Xi 4\ \ (\Psi 42\ \Xi 4)^s, \quad (52)$$

and $\quad (dy_{t+1})^{(3)} = F1\ dy_t + F2\ \xi\varepsilon_{t+1} + P4\ Z4_{t+1}, \quad (53)$

for some matrix $P4$.

(52), (53) can be written as:

$$(dx_{t+1})^{(4)} = Q1\,\xi^2 + Q2 P_2(dy_{t+1}) + Q3\ \xi^2 dy_{t+1} + Q4\ P_3(dy_{t+1}) + Q5\xi^4 + Q6\ \xi^2 P_2(dy_t) + Q7 P_4(dy_t), \quad (54)$$

$$(dy_{t+1})^{(4)} = F1\ dy_t + F2\ \xi\varepsilon_{t+1} + F3\,\xi^2 + F4\,P_2((dy_t;\xi\varepsilon_{t+1}) + F5\,\xi^2\,(dy_t;\xi\varepsilon_{t+1}) + F6\,P_3((dy_t;\xi\varepsilon_{t+1}) + F7\,\xi^4 + F8\,\xi^2 P_2((y_t;\varepsilon_{t+1})) + F9\ P_4((y_t;\varepsilon_{t+1}). \quad (55)$$

where $Q1, Q2, Q3, Q4, Q4, Q6, Q7, F3, F4, F5, F6, F7, F8$ and $F9$ are matrices/vectors.

2.3. Related approaches

The computational approach used here differs from that of Judd and Guu (1993), Judd and Gaspar (1996), Judd (1998) (and most subsequent papers that compute Taylor expansions of the policy function). These authors obtain the coefficients of an $n-th$ order Taylor expansion of by computing the $1st$ to $n-th$ order (cross-) partial derivatives of $H(\omega_t,\xi) \equiv E_t G(f(\omega_t,\xi\varepsilon_{t+1},\xi),\omega_t,\xi\varepsilon_{t+1})$ with respect to ω_t and ξ, at the steady state. Note that these derivatives all have to equal zero, as $H(\omega_t,\xi)=0 \quad \forall\omega_t, \forall\xi \geq 0$. Thus:

$$\partial^i H(\omega_t,\xi)/\partial(\omega_t,\xi)^i \Big|_{\omega_t=\omega,\,\xi=0} = 0 \qquad \text{for } i = 1,..,n. \quad (56)$$

This gives a system of equations in the (unknown) $1st$ to $n-th$ order (cross-) partial derivatives of the policy function f. These partial derivatives can be obtained sequentially: the intercept of $f(\omega_t,\xi\varepsilon_{t+1},\xi)$ is determined by the condition $G(f(\omega,0,0),\omega,0)=0$; first-order derivatives can be found by considering (56) with $i=1$. (56) with $i=2$ pins down the second order derivatives, etc.

The basic difference between that approach and the approach used here can be illustrated using the following simple static model (see Judd (1998), p.449):

$$g(x,\varepsilon)=0, \quad (57)$$

where x and ε are an endogenous and an exogenous variable, respectively. The solution of that model is given by a function

$$x = f(\varepsilon) \text{ that satisfies } g(f(\varepsilon),\varepsilon)=0 \quad \forall\varepsilon. \quad (58)$$

We are interested in computing a Taylor expansion of f around benchmark value ε_0: $dx = f'\, d\varepsilon + \frac{1}{2} f''\, (d\varepsilon)^2 + \frac{1}{6} f'''\, (d\varepsilon)^2 + \ldots$, with $dx = x - x_0$, $d\varepsilon = \varepsilon - \varepsilon_0$, $g(x_0, \varepsilon_0) = 0$. (All derivatives are evaluated at ε_0).

The Judd approach at determining f', f'', f'' etc. is based on these conditions:

$$\partial^i g(f(\varepsilon), \varepsilon) / \partial \varepsilon^i \Big|_{\varepsilon = \varepsilon_0} = 0 \qquad i = 1, .., k.$$

For example: $g_1 f' + g_2 = 0$, which implies that $f' = -g_2/g_1$; $g_{11}(f')^2 + 2g_{12} f' + g_1 f'' + g_{22} = 0$. Substituting $f' = -g_2/g_1$ into this expression allows to determine f'': $f'' = -[g_{11}(g_2/g_1)^2 - 2g_{12}(g_1/g_2) + g_{22}]$.

The approach adopted here, by contrast, computes a second-order approximation using a (slightly) different procedure: namely a second-order approximations of the squared terms (of second-order Taylor expansion) is computed using the first-order solution. A first-order Taylor expansion of (57) gives $g_1 dx + g_2\, d\varepsilon + R_2 = 0$, which implies that $(dx)^{(1)} = -(g_2/g_1)\, d\varepsilon$. A second-order Taylor expansion gives: $g_1 dx + g_2\, d\varepsilon + \frac{1}{2} g_{11}(dx)^2 + g_{12} dx\, d\varepsilon + \frac{1}{2} g_{22}(d\varepsilon)^2 + R_3 = 0$. As $((dx)^2)^{(2)} = ((dx)^{(1)})^2$ and $(dx\, d\varepsilon)^{(2)} = (dx)^{(1)}\, d\varepsilon$, we have:

$$(dx)^{(2)} = -(g_1)^{-1}[g_2\, d\varepsilon + \tfrac{1}{2} g_{11}((dx)^{(1)})^2 + g_{12}(dx)^{(1)}\, d\varepsilon + \tfrac{1}{2} g_{22}(d\varepsilon)^2].$$

This implies that

$$(dx)^{(2)} = -(g_2/g_1)\, d\varepsilon - \tfrac{1}{2}[g_{11}(g_2/g_1)^2 - 2g_{12}(g_1/g_2) + g_{22}]\, (d\varepsilon)^2 .$$

Thus, the implied first and second derivatives of the policy function are identical to those obtained using the Judd approach.

The approach here is closely related to work by, i.a., Kim and Kim (1999), Woodford (1999), and Woodford and Benigno (2003) who have shown that a second-order accurate evaluation of *conditional and unconditional expected values* of ω_t (in a model of type (1)) can be achieved using a first-order accurate model solution. These methods exploit the fact that a first-order accurate (i.e. linear) approximation of the policy function permits a second order accurate evaluation of the squares and cross-products of the state variables (and thus of the second moments of these variables).[5] However, the methods presented by these authors do not readily permit to compute simulated time series $\{\omega_s\}$ that are second order accurate.

Sutherland (2002) uses a linear approximation of the model to provide a second-order accurate evaluation of the conditional expected value of the time path $\{E_0 \omega_s\}_{s \geq 0}$, given the state of the economy at some date $t = 0$. The paper here adapts and generalizes Sutherland's (2002) approach to compute k-th-order accurate simulated paths $\{\omega_s\}$. [6]

3. Applying the method

[5]These papers focus on the computation of welfare. As the vector ω_t can be specified in such a manner that one of its elements includes the utility level of the agents assumed in the model, that approach is sufficient for computing expected welfare.

[6] After the research here was completed, papers by Schaumburg (2002) and Lombardo and Sutherland (2004) were brought to my attention that present *second*-order accurate solutions based on the same idea.

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