Stationary Rational Bubbles in Non-Linear Business Cycle Models of Closed and Open Economies

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Contributions:
(1) new type of sunspot equilibrium that is shown to exist in standard DSGE models, even when Blanchard & Kahn conditions hold. Key mechanism: non-linearity.
(2) new sunspot generates BOOM-BUST cycles
(3) In multi-country model with complete financial markets: BUBBLES must be PERFECTLY correlated across countries
⇒ Bubbles generate SYNCHRONIZED business cycles
Analytical contribution: non-linear DSGE models have more stationary equilibria than you think!

This paper shows: standard *NON-LINEAR* DSGE models have *MULTIPLE* stationary equilibria, even when the linearized versions of these models have unique solution

⇒ In *non-linear* model: stationary fluctuations *WITHOUT* shocks to TFP, preferences, policy
⇒ Blanchard & Kahn (1980): conditions for existence of unique **stable** solution of **linear(ized)** models are **IRRELEVANT** for non-linear models.
Sunspot equilibria in non-linear models studied here look like ‘BUBBLES’:
• economy may temporarily diverge from steady state;
• with exogenous probability economy later reverts to steady state

BOOM-BUST CYCLE:
• consistent with rational expectations
• ‘rational bubbles’ are stationary
Similarities and important differences with rational bubbles in linear models (Blanchard, 1979)

- Like Blanchard (1979) I focus on models whose linearized versions have unique non-explosive equilibrium

- Key difference: bubbles in non-linear models are STATIONARY

- Blanchard bubbles (linear models): *expected* trajectories *explode* to ±∞
Consider non-linear model with just 1 non-predetermined variable (no exogenous driver)
\[ E_t G(Y_{t+1}, Y_t) = 0 \]

Linearization (around steady state):
\[ E_t y_{t+1} = \lambda \cdot y_t, \quad y_t \equiv Y_t - Y^{SS} \]

Linearized model has unique non-explosive solution iff \(|\lambda| > 1\). Unique solution is: \(y_t = 0\) 
(Blanchard & Kahn (1980), Prop. 1)
$E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1; \quad y_t: \text{scalar jump variable}$

Unique stable solution: $y_t = 0$

**Blanchard (1979)**

Bubble: $y_{t+1} = (\lambda/(1-\pi)) \cdot y_t$ with probability $1 - \pi$

$y_{t+1} = 0$ with probability $\pi$

$$\lim_{s \to \infty} E_t y_{t+s} = \pm \infty \quad \text{if} \quad y_t \neq 0$$

Expected path of bubble diverges to $\pm \infty$

Expected path of bubbles in **non-linear** DSGE described here do NOT diverge to $\pm \infty$
Explosive (expected) trajectories are problematic:
► accuracy of linear model approximations breaks down far from point of approximation; non-negativity & technological feasibility constraints may be violated

Example: with decreasing returns to capital, explosive trajectory of capital & output is INFEASIBLE

⇒ LINEAR APPROXIMATION UNSUITABLE FOR ANALYZING RATIONAL BUBBLES
By contrast: **non-linear** analysis here takes non-negativity constraints, decreasing returns & risk aversion into account.

Decreasing returns & risk aversion generate stabilizing forces that prevent explosive trajectories.

Stationary rational bubbles in **non-linear** models are generally **one-sided** (capital over-accumulation, but no under-accumulation).

[By contrast: Blanchard bubbles in linear models can be positive or negative]
• Rational bubbles in non-linear model can induce fluctuations that are close to deterministic steady state most of the time ⇒ unconditional mean of endogenous variables close to deterministic steady state

• Non-linear DSGE models driven just by stationary bubbles can generate persistent fluctuations of real activity & capture key business cycle stylized facts
Note: Can construct linearized DSGE models with non-explosive sunspot equilibria:

\[ E_t y_{t+1} = \lambda \cdot y_t \text{ need } |\lambda| \leq 1. \Rightarrow y_{t+1} = \lambda \cdot y_t + \epsilon_{t+1} \text{ is stationary solution for any } \{\epsilon_{t+1}\} \text{ with } E_t \epsilon_{t+1} = 0 \]

Needed ingredients:
- Increasing returns, externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999))
- Financial frictions (e.g., Martin and Ventura (2018))
- Overlapping generations (e.g., Woodford (1986), Galí (2018))

Specific assumptions & calibrations that deliver \(|\lambda|<1\) can be debatable & fragile (e.g. in standard OLG model: need \(r \leq g\))

By contrast, paper here argues that very standard DSGE models with \(|\lambda|>1\) can deliver stationary sunspot equilibria, if non-linearities are considered.
Related contributions

- Bacchetta, van Wincoop & Tille “Self-fulfilling Risk Panics” (AER 2012): stylized asset pricing model whose linearized version has unique solution, but non-linear model has multiple equilibria iff sunspot shocks are HETEROSKEDASTIC. My paper highlights importance of heteroskedasticity for multiplicity of equilibria in non-linear DSGE business cycle model.
Holden (2016ab) shows that multiple equilibria emerge when occasionally binding constraints (e.g. ZLB) are integrated into otherwise standard linear model.

- By contrast: my analysis considers FULLY non-linear models.
- All model equations are non-linear
- All relevant non-negativity constraints are imposed.
- Model solutions here are globally accurate.
- Multiple equilibria here have “bubbly” dynamics (different from Holden, 2016ab)
Basic intuition I:

Consider non-linear model with just 1 non-predetermined variable (no exogenous driver)

\[ E_t G(Y_{t+1}, Y_t) = 0 \]

Linearization (around steady state):

\[ E_t y_{t+1} = \lambda \cdot y_t, \quad y_t \equiv Y_t - Y^{SS} \]

Linearized model has unique non-explosive solution iff \( |\lambda| > 1 \). Unique solution is: \( y_t = 0 \) (Blanchard & Kahn (1980), Prop. 1)
• RESULT
When $|\lambda| > 1$, the non-linear model can have stationary sunspot equilibrium

• IDEA

$$E_t G(Y_{t+1}, Y_t) = 0 \iff G(Y_{t+1}, Y_t) = \varepsilon_{t+1} \text{ with } E_t \varepsilon_{t+1} = 0$$

$$\Rightarrow Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1}). \quad \varepsilon_{t+1} : \text{“sunspot shock”}$$

Even if $|\Lambda_Y| > 1$, there may exist process $\{\varepsilon_{t+1}\}$ with $E_t \varepsilon_{t+1} = 0$ such that $\{Y_{t+1}\}$ is stationary.

Note: when white noise $\{\varepsilon_{t+1}\}$ is fed into $Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1})$, then $\{Y_{t+1}\}$ diverges if $|\Lambda_Y| > 1$. 
Key requirements for stationary solution:

- \( Y_{t+1} = \Lambda(Y_t, \varepsilon_{t+1}) \) has to be NON-LINEAR in \( \varepsilon_{t+1} \)
- Distribution of \( \varepsilon_{t+1} \) has to depend on \( Y_t \)

\[
Y_{t+1} \approx \Lambda(Y_t, 0) + \Lambda \varepsilon(Y_t, 0) \cdot \varepsilon_{t+1} + \frac{1}{2} \Lambda \varepsilon \varepsilon(Y_t, 0) \cdot (\varepsilon_{t+1})^2
\]

\[
E_t Y_{t+1} \approx \Lambda(Y_t, 0) + \frac{1}{2} \Lambda \varepsilon \varepsilon(Y_t, 0) \cdot E_t(\varepsilon_{t+1})^2
\]

Let \( E_t(\varepsilon_{t+1})^2 = f(Y_t) \geq 0 \). If \( \Lambda \varepsilon \varepsilon(Y_t, 0) \neq 0 \) then can set \( E_t(\varepsilon_{t+1})^2 = f(Y_t) \) such that \[ |dE_t Y_{t+1}/dY_t | < 1: \]

"MEAN REVERSION"

Example: \( \Lambda_Y(Y_t, 0) > 1, \Lambda \varepsilon \varepsilon(Y_t, 0) < 0 \). Then need \( f'(Y_t) > 0 \) for mean reversion: \( E_t(\varepsilon_{t+1})^2 \) must be increasing in \( Y_t \).
Basic intuition II: RBC model

\[ C_t + K_{t+1} = Y_t; \quad Y_t = F(K_t), \quad F' > 0, \quad F'' < 0 \]

\[ \beta \{ E_t u'(C_{t+1}) / u'(C_t) \} \cdot F'(K_{t+1}) = 1; \quad \text{assume } u'''' > 0 \text{ (CRRA)} \]

Sunspot: assume \( K_{t+1} \uparrow \Rightarrow C_t \downarrow \ u'(C_t) \uparrow, \)

\[ F'(K_{t+1}) \downarrow \text{ Euler eqn requires:} \]

\[ E_t u'(C_{t+1}) = E_t u'(F(K_{t+1}) - K_{t+2}) \uparrow \]

- In deterministic economy: need \( C_{t+1} \downarrow \& K_{t+2} \uparrow \)
  \( K_{t+2} \) has to rise more than \( K_{t+1} \) ! \( \Rightarrow \) K diverges

- With stochastic sunspot: \( K_{t+2} \) random.
  \( u'(C_{t+1}) \) is convex in \( K_{t+2} \) \( \Rightarrow \) if \( Var_t(K_{t+2}) \) rises,
  \( E_t u'(C_{t+1}) \uparrow \Rightarrow E_t K_{t+2} \) can rise less than \( K_{t+1} \) !

\( \Rightarrow \) possibility of mean reversion
TRANSVERSALITY CONDITION (TVC)
Standard DSGE usually assume an infinitely-lived representative agent. Optimality conditions include transversality condition:
\[
\lim_{\tau \to \infty} \beta^\tau E_t u'(C_{t+\tau})K_{t+\tau+1} = 0
\]
TVC + Euler eqn. + static efficiency condit. \(\Rightarrow\) unique equilibrium.

When TVC does not hold: economy is “dynamically inefficient”
THIS PAPER DISREGARDS TVC

- Goal is to establish existence of stationary rational bubbles in non-linear DSGE models
- Explosive bubbles in linear (Blanchard) too violate TVC

JUSTIFICATIONS OF MODELS WITHOUT TVC

Assume that there is no TVC because agents are finitely lived (N periods)
Novel result about OLG economy: Assume: (I) Complete financial market allow efficient risk sharing across all generations alive at dates $t$ and $t+1$ (II) Each generation receives wealth endowment such that consumption by newborns is time-invariant share of aggregate consumption. (Under log-utility: wealth endowment of newborns has to be time-invariant share of total wealth) THEN an ‘aggregate’ Euler equation holds that is identical to the Euler equation of a representative infinitely lived household: 

$$\beta E_t \{u'(C_{t+1})/u'(C_t)\}MPK_{t+1} = 1$$

BUT: there is no TVC in the OLG economy!
OLG structure with efficient intergenerational risk sharing:

justification for macro models that lacks a TVC, but whose other equilibrium conditions are identical to those of standard business cycle models (that assume infinitely lived agents)
Other justifications for disregarding TVC

1) Lansing (2010) disregards the TVC in a Lucas-style asset pricing models with bubbles, arguing that “agents are forward-looking but not to the extreme degree implied by the transversality condition.”

2) In richer models with heterogeneous agents and distortions: equilibrium is not solution of decision problem of representative agent.
Detection of TVC violations in stochastic economies: virtually impossible, even with very long simulation runs (billions of periods):
States with very low consumption might only occur with extremely small probabilities.
Detailed Example I: Long-Plosser RBC model with sunspots

\[ u(C) = \ln(C); \quad C_t + K_{t+1} = Y_t; \quad Y_t = F(K_t) \equiv (K_t)^\alpha, \quad 0 < \alpha < 1 \]

Euler equation:

\[ \beta E_t \{u'(C_{t+1})/u'(C_t)\} \cdot F'(K_{t+1}) = 1 \]

\[ \Rightarrow \beta E_t \{C_t / C_{t+1}\} \cdot \alpha Y_{t+1}/K_{t+1} = 1 \]

\[ \Rightarrow \beta E_t \{(Y_t-K_{t+1})/(Y_{t+1}-K_{t+2})\} \cdot \alpha Y_{t+1}/K_{t+1} = 1 \]

\[ \Rightarrow \alpha \beta \cdot E_t \{(1-K_{t+1}/Y_t)/(1-K_{t+2}/Y_{t+1})\} \cdot Y_t/K_{t+1} = 1 \]

\[ \Rightarrow \alpha \beta \cdot E_t \{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1, \quad Z_t \equiv K_{t+1}/Y_t : \text{ investment/output ratio} \]

Textbook solution: \[ Z_t = \alpha \beta \]
\[ \alpha\beta \cdot E_t \{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1 \]

Linearization around \( Z=\alpha\beta \):

\[ E_t z_{t+1} = \lambda \cdot z_t, \quad z_t \equiv Z_t - Z; \quad \lambda \equiv 1/(\alpha\beta) > 1. \]

\[ \Rightarrow z_t = 0 \text{ is unique non-explosive solution of linearized model.} \]

But: **non-linear model has other stationary solutions.**

\[ \alpha\beta \cdot \{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1 + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0 \]

\[ \Rightarrow Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1+\varepsilon_{t+1}). \]

\( Z_{t+1} \) increasing & strictly concave in \( \varepsilon_{t+1} \).
Fig. 1. Long & Plosser model: investment/output ratio at $t+1$, $Z_{t+1}$, as function of $Z_t$ for $\varepsilon_{t+1} \in \{-0.5; 0; 0.5\}$

$Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha \beta (1/Z_t - 1)/(1 + \varepsilon_{t+1}); \quad \alpha = 0.35, \beta = 0.99$. 
\[ Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha \beta (1/Z_t - 1)/(1 + \varepsilon_{t+1}) \]

- When \( Z_t < \alpha \beta \), the model can hit zero-capital corner solution in later periods \( \Rightarrow \) restrict attention to solutions with \( Z_\tau \in [\alpha \beta, 1) \ \forall \tau \)

- Support of \( \varepsilon_{t+1} \) has to be bounded below: \( \varepsilon_{t+1} \geq -1 + [\alpha \beta/(1-\alpha \beta)] \cdot [1/Z_t - 1] \)

\( \Rightarrow \) distribution of \( \varepsilon_{t+1} \) must depend on \( Z_t \)!

- Let \( \varepsilon_{t+1} \) only takes two values: \( -\varepsilon_t \) and \( \varepsilon_t \cdot \pi_t/(1-\pi_t) \) with probabilities \( \pi_t \) and \( 1-\pi_t \), respectively, \( \varepsilon_t \in [0,1) \) \( \Rightarrow Z_{t+1} \) takes two values:

\[ Z^L_{t+1} \equiv \Lambda(Z_t, -\varepsilon_t) \ \& \ Z^H_{t+1} \equiv \Lambda(Z_t, \varepsilon_t \pi_t/(1-\pi_t)) \) with \( Z^L_{t+1} \leq Z^H_{t+1} \leq 1 \).

- Postulate \( Z^L_{t+1} = f(Z_t) \), with \( \alpha \beta \leq f(Z_t) \leq \Lambda(Z_t, 0) \) for \( Z_t \in [\alpha \beta, 1) \).

Solve \( Z^L_{t+1} = \Lambda(Z_t, -\varepsilon_t) \) for \( \varepsilon_t \) & substitute into \( Z^H_{t+1} = \Lambda(Z_t, \varepsilon_t \pi_t/(1-\pi_t)) \).
Degrees of freedom in modeling sunspot:

- bust investment/GDP ratio, $Z_{t+1}^L$
- conditional probability of bust, $\pi_t$

**Specification I:** $Z_{t+1}^L = \alpha \beta + \Delta$, $\Delta = 0.01$, $\pi = 0.5$

(When $\Delta = 0$, then $Z = \alpha \beta = 0.346$ is absorbing state; thus set $\Delta > 0$)
$Z^L_{t+1} = \alpha \beta + \Delta$, $\Delta = 0.01$, $\pi = 0.5$
Simulated series with const. probability: $\pi=0.5$

Simulated output ($Y$), consumption ($C$) and investment ($I$) normalized by steady state output
Lower volatility if probability of investment bust rises once investment/output ration $Z_t$ crosses threshold.

Simulated series with state-contingent probability of bust:

$$\pi_t = 0.5 \text{ for } \alpha \beta + \Delta = 0.356 \leq Z_t \leq 0.36 \quad \& \quad \pi_t = 1 - 10^{-100} \text{ for } Z_t > 0.36$$
Table 1. Long-Plosser model with bubbles: predicted business cycle statistics

| SS | Standard dev. | Corr. with Y | Autocorr. | Mean (% deviation from | |
|----|---------------|--------------|-----------|------------------------| |
|    | Y  C  I      | C  I         | Y  C  I   | Z                      | |
| (1)| (2) (3)      | (4) (5)      | (6) (7) (8)| (9) (10) (11) (12)     | |
| (a) Specification I: $Z_t^L = αβ + Δ$

| $π_t=0.5$ | 11.72 100.19 33.48 | -0.42 0.62 | 0.62 0.47 0.62 | 13.49 -7.62 53.31 31.15 | |
| $π_t≥1$ for $z_t>0.36$ | 1.33 3.51 3.82 | 0.77 -0.26 | -0.26 -0.66 -0.26 | 3.27 -0.13 9.71 6.25 | |
| (b) US Data (from King and Rebelo (1999))

| 1.81 | 1.35 | 5.30 | 0.88 | 0.80 | 0.88 | 0.80 | 0.87 |

Note: all business statistics pertain to HP-filtered logged variables.
Example II: RBC model with incomplete capital depreciation & endogenous labor

\[ U(C_t, L_t) = \ln(C_t) + \Psi \cdot \ln(1-L_t), \quad \Psi > 0, \quad L_t : \text{hours worked} \]

\[ C_t + K_{t+1} = Y_t + (1-\delta)K_t , \quad Y_t = \theta(K_t)^{\alpha} (L_t)^{1-\alpha} \]

- **FOCs:**
  \[ C_t \Psi / (1-L_t) = (1-\alpha)\theta(K_t)^{\alpha} (L_t)^{-\alpha} \]

  \[ E_t \beta \{ C_t / C_{t+1} \} (\alpha \theta(K_{t+1})^{\alpha-1} (L_{t+1})^{1-\alpha} + 1-\delta) = 1 \]

- Using static efficiency conditions can express \( C \) & \( L \) as functions of capital:
  \[ C_t = \gamma(K_{t+1}, K_t), \quad L_t = \eta(K_{t+1}, K_t) \]

Can write Euler equation as:

\[ E_t[\beta \{ \gamma(K_{t+1}, K_t) / \gamma(K_{t+2}, K_{t+1}) \} (\alpha \theta(K_{t+1})^{\alpha-1} (\eta(K_{t+2}, K_{t+1}))^{1-\alpha} + 1-\delta)] = 1 \]
Euler equation:

\[ E_t H(K_{t+2}, K_{t+1}, K_t) = 1 \]

No bubble solution (TVC): described by policy function \( K_{t+1} = \lambda(K_t) \)

so that \( E_t H(\lambda(\lambda(K_t)), \lambda(K_t), K_t) = 1 \)
Consider bubble equilibria such that, for any $t$, $K_{t+1}$ takes one of two values $K_{t+1} \in \{K^L_{t+1}, K^H_{t+1}\}$ with exogenous probabilities $\pi$ and $1-\pi$, where $K^L_{t+1} = \lambda(K_t)e^\Delta$;

$\Delta > 0$: small positive constant

‘L’ is ‘bust’ state, in which capital stock set at $t$ reverts to value close to ‘no-bubble’ decision rule

Euler equation

$E_t H(K_{t+2}, K_{t+1}, K_t) = 1$

becomes:

$\pi H(\lambda(K_{t+1})e^\Delta, K_{t+1}, K_t) + (1-\pi) \cdot H(K^H_{t+2}, K_{t+1}, K_t) = 1$
Economy evolves as follows:

At date t: random draw (with probab. \( \pi, 1-\pi \)) determines \( K_{t+1} \in \{K_L^{t+1}, K_H^{t+1}\} \) where \( K_L^{t+1} = \lambda(K_t)e^\Delta \)

Euler equation between t and t+1 determines \( K_H^{t+2} \):

\[
\pi H(\lambda(K_{t+1})e^\Delta, K_{t+1}, K_t) + (1-\pi) \cdot H(K_H^{t+2}, K_{t+1}, K_t) = 1
\]

Etc. in all subsequent periods.

See paper for: ● Existence proof of sunspot equilibrium: need \( \Delta > 0 \). Then \( K_L^{t+1} < K_H^{t+1} \)
● Analysis with stochastic TFP

Numerical simulations

\( \beta = 0.99; \ \alpha = \frac{1}{3}; \ \delta = 0.025; \)

Labor supply elasticity (at steady state) = 1.

● Log utility (unit risk aversion, RA): \( \ln(C_t) \)

● ‘High Risk Aversion’ utility: \( \ln(C_t - \overline{C}) \), \( \overline{C} > 0 \)

Parameters of bubble process:

\( \Delta = 0.001 \)

Bust probability: \( \pi = 0.5, \ \nu = 0.2. \)
RBC model (incomplete capital deprec.)
with bubbles: predicted business cycle statistics

<table>
<thead>
<tr>
<th>Unit Risk aversion</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>π=0.5   π=0.2</td>
<td></td>
</tr>
<tr>
<td>(1)     (2)</td>
<td>(5)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviations [in %]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>0.49</td>
<td>1.16</td>
</tr>
<tr>
<td>C</td>
<td>1.08</td>
<td>2.63</td>
</tr>
<tr>
<td>I</td>
<td>4.29</td>
<td>9.38</td>
</tr>
<tr>
<td>L</td>
<td>0.74</td>
<td>1.73</td>
</tr>
</tbody>
</table>

|                      |     |     |     |
| Correlations with GDP|     |     |     |
| C                    | -0.97 | -0.95 | -0.99 | -0.98 | 0.88 |
| I                    | 0.98  | 0.96  | 0.99  | 0.99  | 0.80 |
| L                    | 0.99  | 0.97  | 0.99  | 0.99  | 0.88 |

|                      |     |     |     |     |
| Autocorrelations    |     |     |     |     |
| Y                    | 0.36 | 0.63 | 0.35 | 0.62 | 0.84 |
| C                    | 0.33 | 0.60 | 0.35 | 0.62 | 0.80 |
| I                    | 0.36 | 0.63 | 0.37 | 0.64 | 0.87 |
| L                    | 0.34 | 0.61 | 0.35 | 0.62 | 0.88 |

|                      |     |     |     |     |
| Means [% deviation from steady state] |     |     |     |     |
| Y                    | 1.41 | 2.80 | 1.25 | 2.12 | -- |
| C                    | 0.73 | 1.39 | 0.33 | 0.55 | -- |
| I                    | 3.62 | 7.33 | 4.22 | 7.19 | -- |
| L                    | 0.36 | 0.74 | -0.02| -0.02| -- |

|                      |     |     |     |     |
| Mean (capital income – investment)/GDP [in %] |     |     |     |     |
| 9.12 8.75 8.93 8.54 13.42 |

|                      |     |     |     |     |
| Fraction of periods with (capital income > investment) [in %] |     |     |     |     |
| 99.2 96.3 99.5 97.7 100 |
Non-linear RBC model (incompl. capital depreciation) driven by bubbles
Simulated GDP, C and I series normalized by steady state GDP. Hours worked (L) normalized by steady state hours.
Example III: Dellas (1986) 2-country RBC = 2-country version of Long & Plosser

- Countries: $i=H,F$ (‘Home’ & ‘Foreign’)
- Each country specialized in production of distinct tradable intermediate good
  
  \[ Y_{i,t} = \theta_{i,t} (K_{i,t})^\alpha, \quad K_{i,t+1} = I_{i,t} \quad \text{(full capital depreciation)} \]

- Each country uses local & imported intermediates to produce a non-tradable final good used for consumption & investment:
  
  \[ Z_{i,t} = (y_{i,t}^i/\xi)^\xi \cdot (y_{i,t}^i/(1-\xi))^{1-\xi}, \quad i \neq j. \quad Z_{i,t} : \text{final good,} \]
  \[ Z_{i,t} = C_{i,t} + K_{i,t+1} \]
  \[ y_{i,t}^j \quad \text{amount of input } j \text{ used by country } i \]
  \[ \frac{1}{2} < \xi < 1 : \quad \text{local spending bias} \]
- Final good price: $Z_{i,t} = (p_{i,t})^\xi \cdot (p_{j,t})^{1-\xi}$
- Demand for local & imported intermediates:
  \[ p_{i,t}y_{i,t}^i = \xi \cdot P_{i,t} \cdot (C_{i,t} + K_{i,t+1}), \quad p_{j,t}y_{i,t}^j = (1-\xi) \cdot P_{i,t} \cdot (C_{i,t} + K_{i,t+1}), \quad i \neq j. \]
- Log utility: $U_{i,t} = \ln(C_{i,t})$
- Complete international financial markets: full risk sharing: $P_{H,t}C_{H,t} = P_{F,t}C_{F,t}$
- Market clearing for intermediates:
  \[ Y_{H,t} = y_{H,t}^H + y_{F,t}^H, \quad Y_{F,t} = y_{H,t}^F + y_{F,t}^F \quad \Rightarrow \]
  \[ p_{H,t}Y_{H,t} = \xi \cdot (P_{H,t}C_{H,t} + P_{H,t}K_{H,t+1}) + (1-\xi) \cdot (P_{F,t}C_{F,t} + P_{F,t}K_{F,t+1}). \]
  \[ p_{F,t}Y_{F,t} = (1-\xi) \cdot (P_{H,t}C_{H,t} + P_{H,t}K_{H,t+1}) + \xi \cdot (P_{F,t}C_{F,t} + P_{F,t}K_{F,t+1}). \]
- Market clearing:
  \[ p_{H,t}Y_{H,t} = p_{H,t}C_{H,t} + \xi \cdot p_{H,t}K_{H,t+1} + (1-\xi) \cdot p_{F,t}K_{F,t+1} \]
  \[ p_{F,t}Y_{F,t} = p_{F,t}C_{F,t} + (1-\xi) \cdot p_{H,t}K_{H,t+1} + \xi \cdot p_{F,t}K_{F,t+1} \]

- Euler equation:
  \[ E_t \beta(C_{i,t}/C_{i,t+1}) \alpha p_{i,t+1}Y_{i,t+1}/K_{i,t+1} = 1 \]
  \[ \Rightarrow \alpha \beta E_t \{(P_{i,t}C_{i,t})/(P_{i,t+1}C_{i,t+1})\}(p_{i,t+1}Y_{i,t+1})/(P_{i,t}K_{i,t+1}) = 1 \]

- \[ \kappa_{i,t} \equiv P_{i,t}K_{i,t+1}/(P_{i,t}C_{i,t}) \]
  \[ \Rightarrow \alpha \beta E_t \{1 + \xi \kappa_{i,t+1} + (1-\xi)\kappa_{j,t+1}\} = \kappa_{i,t} \]

\( \frac{1}{2} < \xi < 1 \) spending home bias
\[
\begin{bmatrix}
E_t K_{H,t+1} \\
E_t K_{F,t+1}
\end{bmatrix} = c + B \begin{bmatrix}
K_{H,t} \\
K_{F,t}
\end{bmatrix}, \text{ with } B = \frac{1}{\alpha \beta (2 \xi - 1)} \begin{bmatrix}
\xi & -(1 - \xi) \\
-(1 - \xi) & \xi
\end{bmatrix}.
\]

\text{eig}(B) > 1

- Strictly positive process for Home and Foreign investment/consumption requires
  \[K_{H,t} = K_{F,t} \geq 0\]

**Bubble perfectly correlated across countries**

- Investment bubble only in Home country:
  \[\Rightarrow\] growing Home trade deficit
  \[\Rightarrow\] would put Foreign investment on a downward trajectory.
  \[\Rightarrow\] Foreign capital could ultimately reach zero.
Logic of global bubbles also holds in BKK IRBC

(1) Bubbles, no TFP shocks.

(2) Just TFP shocks (no bubble)

(3) Bubbles & TFP shocks
CONCLUSIONS

- Stationary sunspot equilibria exist in standard non-linear DSGE models, even when the linearized versions of those models have unique solutions.
- In the sunspot equilibria considered here, the economy temporarily diverges from the no-sunspots trajectory, before abruptly reverting towards that trajectory.
- In multi-country model: bubble perfectly correlated across countries