Rational Bubbles in Non-Linear Business Cycle Models: Closed and Open Economies

Robert Kollmann
Université Libre de Bruxelles & CEPR

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Contribution: construct BOUNDED “RATIONAL BUBBLES” in standard NONLINEAR RBC macro models

Rational bubbles (Blanchard, 1979): multiple equilibria arising from absence of transversality condition (TVC) for asset

► Blanchard’s rational bubble pertains to model in which TVC pins down unique stable saddle path.
► Rational bubble (no TVC) imply that economy deviates from saddle path, but may stochastically revert to saddle path: Booms followed by sudden busts
• Lack of TVC can be due to OLG structure with finitely-lived households

• Key contribution: without TVC, the non-linear models here have multiple bounded solutions

BOUNDINESS reflects NON-LINEAR EFFECTS
• Bounded rational bubbles feature **recurrent boom-bust cycles** in investment & output: persistent expansions followed by abrupt contractions

Rational bubbles reflect self-fulfilling fluctuations in agents’ expectations about future investment

• Bounded rational bubbles in non-linear models: novel driver of business cycles

Non-linear DSGE models driven just by stationary bubbles can generate **persistent fluctuations of real activity** & capture **key business cycle stylized facts**
• **BOUNDEDNESS** of rational bubbles reflects **NON-LINEAR EFFECTS**: Rational bubbles in LINEARIZED models are **explosive**: expected trajectories tend to $\pm \infty$

• But: rational bubbles in NON-LINEAR models can be bounded (stable)

$\Rightarrow$ Blanchard & Kahn (1980): conditions for existence of unique *stable* solution of linear(ized) models are **IRRELEVANT** for non-linear models

Non-linear DSGE models have more stationary solutions than you think!
Blanchard (1979): rational bubbles in linear asset pricing models without TVC

\[ E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1; \quad y_t: \text{scalar jump variable} \]

Unique stable solution: \( y_t = 0 \)

Blanchard (1979)

Bubble: \( y_{t+1} = (\lambda/(1-\pi)) \cdot y_t \) with probability \( 1 - \pi \)

\[ y_{t+1} = 0 \quad \text{with probability } \pi \]

\[ \lim_{s \to \infty} E_t y_{t+s} = \pm \infty \text{ if } y_t \neq 0 \]

expected path of bubble diverges to \( \pm \infty \)
Blanchard (1979): powerful narrative, much cited (mainly in finance), but NO influence on structural macro modeling.

This paper: shows how to embed Blanchard (1979) bubble into DSGE model

Key contribution: in non-linear model BUBBLES CAN BE BOUNDED

Expected path of bubbles in non-linear DSGE studied here do NOT diverge to ±∞
WHY BOUNDEDNESS OF RATIONAL BUBBLES IS IMPORTANT:

- Explosive bubbles trajectories are problematic:
  - accuracy of linear(ized) models breaks down far from point of approximation; non-negativity & technological feasibility constraints may be violated

Example: with decreasing returns to capital, explosive trajectory of capital & output is INFEASIBLE
LINEARIZED MODELS UNSUITABLE FOR ANALYZING RATIONAL BUBBLES

• By contrast: non-linear analysis here takes non-negativity constraints, decreasing returns & risk aversion into account
• Decreasing returns & risk aversion generate stabilizing forces that prevent explosive trajectories
• Stationary rational bubbles in non-linear models are one-sided (capital over-accumulation, but no under-accumulation)

[By contrast: Blanchard bubbles in linear models can be positive or negative]
• Rational bubbles in non-linear model can induce fluctuations that are close to deterministic steady state most of the time
⇒ unconditional mean of endogenous variables close to deterministic steady state
Note: Can construct linearized DSGE models with non-explosive sunspot equilibria:

\[ E_t y_{t+1} = \lambda \cdot y_t \text{ need } |\lambda| \leq 1. \Rightarrow y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1} \text{ is stationary solution for any } \{\varepsilon_{t+1}\} \text{ with } E_t \varepsilon_{t+1} = 0 \]

Needed ingredients:
- Increasing returns, externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999))
- Financial frictions (e.g., Martin and Ventura (2018))
- Overlapping generations (e.g., Woodford (1986), Galí (2018))

Specific assumptions & calibrations that deliver $|\lambda| < 1$ can be debatable & fragile (e.g. in standard OLG model: need $r \leq g$)

*By contrast, paper here argues that very standard DSGE models with $|\lambda| > 1$ can deliver stationary sunspot equilibria, if non-linearities are considered.*
Example I:
Long-Plosser RBC model with ration. bubble

\[ u(C) = \ln(C) \]; \( C_t + K_{t+1} = Y_t \); \( Y_t = F(K_t) \equiv (K_t) \alpha \), \( 0 < \alpha < 1 \)

Euler equation:

\[ \beta E_t \{ u'(C_{t+1})/u'(C_t) \} \cdot F'(K_{t+1}) = 1 \]
\[ \Rightarrow \beta E_t \{ C_t / C_{t+1} \} \cdot \alpha Y_{t+1}/K_{t+1} = 1 \]
\[ \Rightarrow \beta E_t \{ (Y_t - K_{t+1})/(Y_{t+1} - K_{t+2}) \} \cdot \alpha Y_{t+1}/K_{t+1} = 1 \]
\[ \Rightarrow \alpha \beta \cdot E_t \{ (1 - K_{t+1}/Y_t)/(1 - K_{t+2}/Y_{t+1}) \} \cdot Y_t/K_{t+1} = 1 \]
\[ \Rightarrow \alpha \beta \cdot E_t \{ (1 - Z_t)/(1 - Z_{t+1}) \}/Z_t = 1, \quad Z_t \equiv K_{t+1}/Y_t : \text{investment/output ratio} \]

Textbook solution: \( Z_t = \alpha \beta \)
$$\alpha \beta \cdot E_t \left\{ \frac{(1-Z_t)}{(1-Z_{t+1})} \right\}/Z_t = 1$$

Linearization around $Z = \alpha \beta$:

$$E_t Z_{t+1} = \lambda \cdot z_t, \quad z_t = Z_t - Z; \quad \lambda = 1/(\alpha \beta) > 1.$$ 

$\Rightarrow z_t = 0$ is **unique** non-explosive solution of linearized model.

But: **non-linear** model has other stationary solutions.

$$\alpha \beta \cdot \left\{ \frac{(1-Z_t)}{(1-Z_{t+1})} \right\}/Z_t = 1 + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

$\Rightarrow Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha \beta (1/Z_t - 1)/(1+\varepsilon_{t+1}).$

$Z_{t+1}$ increasing & strictly concave in $Z_t$ & $\varepsilon_{t+1}$.
Fig. 1. Long & Plosser model: investment/output ratio at $t+1$, $Z_{t+1}$, as function of $Z_t$ for $\varepsilon_{t+1} \in \{-0.5; 0; 0.5\}$

$$Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) = 1 - \alpha\beta(1/Z_t - 1)/(1 + \varepsilon_{t+1}); \quad \alpha = 1/3, \beta = 0.99.$$
• Intuition why $Z_{t+1}$ is increasing in $Z_t$:

$C_t + K_{t+1} = Y_t; \quad Y_t = F(K_t), \quad F' > 0, \quad F'' < 0$

$\beta \{E_t u'(C_{t+1})/u'(C_t)\} \cdot F'(K_{t+1}) = 1; \quad u''' > 0 \quad \text{(CRRA)}$

Sunspot: assume $K_{t+1} \uparrow \Rightarrow C_t \downarrow \quad u'(C_t) \uparrow,$

$F'(K_{t+1}) \downarrow \quad \text{Euler eqn requires:}$

$E_t u'(C_{t+1}) = E_t u'(F(K_{t+1}) - K_{t+2}) \uparrow$

• In deterministic economy: need $C_{t+1} \downarrow \quad \& K_{t+2} \uparrow$

$K_{t+2}$ has to rise more than $K_{t+1}! \Rightarrow K \text{ diverges}$

Thus: $Z_t \uparrow \Rightarrow Z_{t+1} \uparrow$
\[ Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta \left( \frac{1}{Z_t} - 1 \right)/(1 + \varepsilon_{t+1}) \]

- When \( Z_t < \alpha\beta \), the model can hit zero-capital corner solution in later periods ⇒ restrict attention to solutions with \( Z_\tau \in [\alpha\beta, 1) \ \forall \tau \)

- Support of \( \varepsilon_{t+1} \) has to be bounded below: \( \varepsilon_{t+1} \geq -1 + [\alpha\beta/(1 - \alpha\beta)] \cdot [1/Z_t - 1] \) ⇒ distribution of \( \varepsilon_{t+1} \) must depend on \( Z_t \)!

- Let \( \varepsilon_{t+1} \) only takes two values: \( -\varepsilon_t \) and \( \varepsilon_t \cdot \pi_t/(1 - \pi_t) \) with probabilities \( \pi_t \) and \( 1 - \pi_t \), respectively, \( \varepsilon_t \in [0,1) \) ⇒ \( Z_{t+1} \) takes two values:
  \[ Z_{t+1}^L \equiv \Lambda(Z_t, -\varepsilon_t) \ \& \ Z_{t+1}^H \equiv \Lambda(Z_t, \varepsilon_t \pi_t/(1 - \pi_t)) \] with \( Z_{t+1}^L \leq Z_{t+1}^H \leq 1 \).
Specification à la Blanchard (1979): Given $Z_t^L$, next period two values possible:

- $Z_{t+1}^L = \alpha \beta + \Delta$, $\Delta = 10^{-6}$ with probability $\pi = 0.5$
  (When $\Delta = 0$, then $Z = \alpha \beta = 0.346$ is absorbing state; thus set $\Delta > 0$)

- $Z_{t+1}^H > \alpha \beta + \Delta$ with prob. $1 - \pi$

$Z_{t+1}^H$ pinned down by Euler eqn:

$$
\pi \cdot \alpha \beta \cdot \{(1-Z_t)/(1-Z_{t+1}^L)\}/Z_t + (1-\pi) \cdot \alpha \beta \cdot \{(1-Z_t)/(1-Z_{t+1}^H)\}/Z_t = 1
$$
Simulated series with const. probability: $\pi=0.5$
Simulated output (Y), consumption (C) and investment (I) normalized by steady state output
Table 1. Long-Plosser model (closed economy) with bubbles: business cycle statistics

<table>
<thead>
<tr>
<th></th>
<th>Standard dev. %</th>
<th>Corr. with Y</th>
<th>Autocorrelations</th>
<th>Mean [% deviation from SS]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>C</td>
<td>I</td>
<td>C</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
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<td>(4)</td>
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<tr>
<td>(a) Predicted business cycle statistics</td>
<td>1.14</td>
<td>2.35</td>
<td>3.41</td>
<td>0.38</td>
</tr>
<tr>
<td>(b) Historical business cycle statistics</td>
<td>1.47</td>
<td>1.19</td>
<td>4.96</td>
<td>0.87</td>
</tr>
</tbody>
</table>

Notes: Row (a) reports simulated business cycle statistics for a Long-Plosser economy with bubbles (no transversality condition); see Sect. 2 of paper. Y: output; C: consumption; I: investment; Z: investment/output ratio.
Novel result about OLG economy:
Assume: (I) Complete financial market allow efficient risk sharing across all generations alive at dates \( t \) and \( t+1 \)
(II) Each generation receives wealth endowment such that consumption by newborns is time-invariant share of aggregate consumption. (Under log-utility: wealth endowment of newborns has to be time-invariant share of total wealth)

THEN
an ‘aggregate’ Euler equation holds that is identical to the Euler equation of a representative infinitely lived household:
\[
\beta E_t \left\{ \frac{u'(C_{t+1})}{u'(C_t)} \right\} MPK_{t+1} = 1
\]
BUT: there is no TVC in the OLG economy!
OLG structure with efficient intergenerational risk sharing:

justification for macro models that lacks a TVC, but whose other equilibrium conditions are identical to those of standard business cycle models (that assume infinitely lived agents)
Example II: RBC model with incomplete capital depreciation & endogenous labor

\[ U(C_t, L_t) = \ln(C_t) + \Psi \cdot \ln(1-L_t), \quad \Psi > 0, \quad L_t : \text{hours worked} \]

\[ C_t + K_{t+1} = Y_t + (1-\delta)K_t, \quad Y_t = \theta(K_t)^\alpha (L_t)^{1-\alpha} \]

- **FOCs:**
  \[ C_t \Psi / (1-L_t) = (1-\alpha)\theta(K_t)^\alpha (L_t)^{-\alpha} \]
  \[ E_t \beta \{ C_t / C_{t+1} \} (\alpha \theta(K_{t+1})^{\alpha-1} (L_{t+1})^{1-\alpha} + 1-\delta) = 1 \]

- Using static efficiency conditions can express \( C \) & \( L \) as functions of capital:
  \[ C_t = \gamma(K_{t+1}, K_t), \quad L_t = \eta(K_{t+1}, K_t) \]

Can write Euler equation as:
\[ E_t \left[ \beta \{ \gamma(K_{t+1}, K_t) / \gamma(K_{t+2}, K_{t+1}) \} (\alpha \theta(K_{t+1})^{\alpha-1} (\eta(K_{t+2}, K_{t+1}))^{1-\alpha} + 1-\delta) \right] = 1 \]
Euler equation:
\[ E_t H(K_{t+2}, K_{t+1}, K_t) = 1 \]

No bubble solution (TVC): described by policy function \( K_{t+1} = \lambda(K_t) \)
so that \( E_t H(\lambda(\lambda(K_t)), \lambda(K_t), K_t) = 1 \)
Consider **bubble equilibria** such that, for any \( t \), \( K_{t+1} \) takes one of two values \( K_{t+1} \in \{ K^L_{t+1}, K^H_{t+1} \} \) with exogenous probabilities \( \pi \) and \( 1-\pi \), where \( K^L_{t+1} = \lambda(K_t) e^\Delta \); \( \Delta > 0 \): small positive constant

‘L’ is ‘bust’ state, in which capital stock set at \( t \) reverts to value close to ‘no-bubble’ decision rule

Euler equation
\[
E_t H(K_{t+2}, K_{t+1}, K_t) = 1
\]
becomes:
\[
\pi H(\lambda(K_{t+1}) e^\Delta, K_{t+1}, K_t) + (1-\pi) \cdot H(K^H_{t+2}, K_{t+1}, K_t) = 1
\]
Economy evolves as follows:

At date $t$: random draw (with probab. $\pi, 1-\pi$) determines $K_{t+1} \in \{K^L_{t+1}, K^H_{t+1}\}$ where $K^L_{t+1} = \lambda(K_t)e^\Delta$

Euler equation between $t$ and $t+1$ determines $K^H_{t+2}$:

$$\pi H(\lambda(K_{t+1})e^\Delta, K_{t+1}, K_t) + (1-\pi) \cdot H(K^H_{t+2}, K_{t+1}, K_t) = 1$$

Etc. in all subsequent periods.

See paper for: ● Existence proof of sunspot equilibrium: need $\Delta > 0$. Then $K^L_{t+1} < K^H_{t+1}$
Analysis with stochastic TFP

Numerical simulations

$\beta=0.99; \quad \alpha=1/3; \quad \delta=0.025$;

Labor supply elasticity (at steady state) = 1.

- Log utility (unit risk aversion, RA): $\ln(C_t)$
- ‘High Risk Aversion’ utility: $\ln(C_t-C), \overline{C}>0$

Parameters of bubble process:

$\Delta=0.001$

Bust probabilities: $\pi=0.5, \pi'=0.2$. 
<table>
<thead>
<tr>
<th>Unit Risk aversion</th>
<th>High RA</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi=0.5) (\pi=0.2) (\pi=0.5) (\pi=0.2)</td>
<td>(1) (2) (3) (4) (5)</td>
<td></td>
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<tr>
<td>Standard deviations [in %]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>0.49</td>
<td>1.16</td>
</tr>
<tr>
<td>C</td>
<td>1.08</td>
<td>2.63</td>
</tr>
<tr>
<td>I</td>
<td>4.29</td>
<td>9.38</td>
</tr>
<tr>
<td>L</td>
<td>0.74</td>
<td>1.73</td>
</tr>
<tr>
<td>Correlations with GDP</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>-0.97</td>
<td>-0.95</td>
</tr>
<tr>
<td>I</td>
<td>0.98</td>
<td>0.96</td>
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<tr>
<td>Y</td>
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<tr>
<td>C</td>
<td>0.33</td>
<td>0.60</td>
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<tr>
<td>I</td>
<td>0.36</td>
<td>0.63</td>
</tr>
<tr>
<td>L</td>
<td>0.34</td>
<td>0.61</td>
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<tr>
<td>Means [% deviation from steady state]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>1.41</td>
<td>2.80</td>
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<tr>
<td>C</td>
<td>0.73</td>
<td>1.39</td>
</tr>
<tr>
<td>I</td>
<td>3.62</td>
<td>7.33</td>
</tr>
<tr>
<td>L</td>
<td>0.36</td>
<td>0.74</td>
</tr>
<tr>
<td>Mean (capital income – investment)/GDP [in %]</td>
<td></td>
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<tr>
<td></td>
<td>9.12</td>
<td>8.75</td>
</tr>
<tr>
<td>Fraction of periods with (capital income &gt; investment) [in %]</td>
<td></td>
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<tr>
<td></td>
<td>99.2</td>
<td>96.3</td>
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</tbody>
</table>
Non-linear RBC model (incompl. capital depreciation) driven by bubbles
Simulated GDP, C and I series normalized by steady state GDP. Hours worked (L) normalized by steady state hours.
CONCLUSIONS

● Bounded rational bubbles exist in standard non-linear DSGE models, even when the linearized versions of those models have unique solutions.
● Rational bubbles: novel source of business cycles
● Induce boom-bust cycles: persistent investment and output expansions followed by abrupt contractions